

How to construct generalized logistic curves associated with the complex logistic equation:

An application to COVID-19

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complex logistic equation
pole point
generalized logistic curve
COVID-19

▼ Summary

This paper, first, reviews the complex logistic equation and its solution. Then, we define the generalized logistic curve and study its properties. Furthermore, we approximate the variation of the number of infected patients of COVID-19 by using the generalized logistic curve.

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Introduction

In order to study population dynamics, T.R.Malthus introduced the differential equation:

$$(0.1) \quad \frac{dx}{dt} = ax$$

where a is a positive constant ([2]). Equation (0.1) is known as the Malthus growth equation. Furthermore, P.F.Verhulst modified Equation (0.1) to Equation (0.2):

$$(0.2) \quad \frac{dx}{dt} = ax - bx^2$$

where $a > 0$ and $b > 0$ ([3]). Equation (0.2) is known as the logistic equation.

In our previous paper ([1]), we introduced the complex logistic equation:

$$(0.3) \quad \frac{dZ}{dT} = aZ(1-Z)$$

where a , T and Z are complex numbers, and using the solution of Equation (0.3), we constructed the generalized logistic curve. In this paper, we will express the detailed construction of generalized logistic curves associated with the complex logistic equation. Presumably, using these generalized logistic curves, we can depict the variation of the number of infected patients of COVID-19.

This paper consists of four sections, as follows. The first section reviews the complex logistic equation. The second section constructs generalized logistic curves. The third section illustrates generalized logistic curves. The last section depicts the variation of the number of infected patients of COVID-19 by the generalized logistic curve.

1 A review of the complex logistic equation

In this section, we will review the complex logistic equation according to [1].

Let a be a non-zero complex number. The equation:

$$(1.1) \quad \frac{dZ}{dT} = aZ(1-Z)$$

with the initial condition $z_0 \neq 0, 1$ has the explicit solution:

$$(1.2) \quad Z(T) = \frac{1}{\frac{1-z_0}{z_0}e^{-aT} + 1}$$

Using the exponential expression $\frac{1-z_0}{z_0} e^{u_0+v_0\sqrt{-1}}$ and the coordinate transformation of $aT = U = u + v\sqrt{-1}$, Equation (1.2) is rewritten by U as follows:

$$(1.3) \quad \tilde{Z}(U) = \frac{1}{e^{(u_0-u)} \cdot e^{(v_0-v)\sqrt{-1}} + 1}$$

We consider the absolute value squared of $\tilde{Z}(U)$:

$$(1.4) \quad |\tilde{Z}(U)|^2 = \frac{1}{e^{2(u_0-u)} + 2e^{(u_0-u)} \cdot \cos(v_0-v) + 1}$$

We envisage that, regarding the two real variables u and v in Equation (1.4), the former represents a change of population size and the latter represents an environmental variable such as the change of the seasons, etc.

When a point $U = (u, v)$ in $\mathbf{R}^2$ moves along the line L , we define the function $\varphi(\theta)$ of θ as the restriction of $|\tilde{Z}(U)|^2$ on L . We call this graph of $\varphi(\theta)$ the generalized logistic curve.

In the next section, we will give details on the construction of $\varphi(\theta)$.

2 Construction of $\varphi(\theta)$

Let U be a point on the plane $\mathbf{R}^2$. U is represented (u, v) by the coordinate of $\mathbf{R}^2$. Take the point $U_0 = (u_0, v_0) \in \mathbf{R}^2$. We denote by A the set of pole points of $\tilde{Z}(U)$ which is defined by

$$(2.1) \quad \begin{aligned} A &= \{u_0 - u = 0 \text{ and } v_0 - v = \pi + 2\pi k \mid k \in \mathbf{Z}\} \\ &= \{A_k = (u_0, v_0 - \pi - 2\pi k) \mid k \in \mathbf{Z}\} \end{aligned}$$

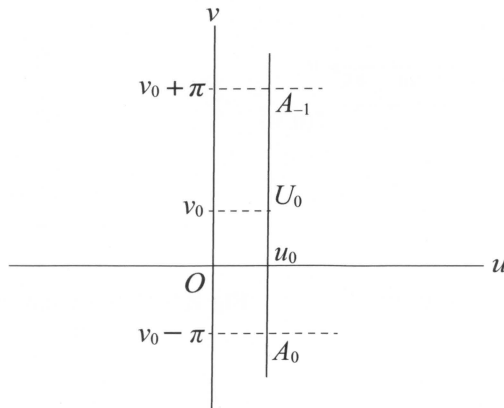


Fig 2.1: U -Plane $((u, v)$ -plane)

We note that following properties are clear:

Periodic property: $\tilde{Z}(u, v + 2\pi) = \tilde{Z}(u, v)$

Symmetric property: $\tilde{Z}(u, v_0 + v) = \tilde{Z}(u, v_0 - v)$.

We denote by $\vec{g}_\lambda = \frac{1}{\sqrt{1+\lambda^2}} (1, \lambda)$ a gradient vector where λ is a real number. Then we define the line $L(\lambda, \xi)$ with the gradient vector $\vec{g}_\lambda$ and the initial point $(u_0, v_0 + \xi)$ as follows:

$$(2.2) \quad L(\lambda, \xi): U = \theta \cdot \vec{g}_\lambda + (u_0, v_0 + \xi)$$

where θ and ξ are real numbers.

We denote by $\varphi_{\lambda, \xi}(\theta)$ the restriction of $|\tilde{Z}(U)|^2$ on $L(\lambda, \xi)$:

$$(2.3) \quad \varphi_{\lambda, \xi}(\theta) = \frac{1}{e^{-2\theta \frac{1}{\sqrt{1+\lambda^2}}} + 2e^{-\theta \frac{1}{\sqrt{1+\lambda^2}}} \cos(\theta \frac{\lambda}{\sqrt{1+\lambda^2}} + \xi) + 1}$$

Now, to understand the function $\varphi_{\lambda, \xi}(\theta)$, we give the well-known example, that is, it is known as the logistic curve in the sense of Verhulst.

Example 2.1 If α is equal to $1.0 + 0.5\sqrt{-1}$ and $z_0 = 0.2 + 0.3\sqrt{-1}$, we obtain $U_0 = (u_0, v_0) = (0.86, -1.34)$ by using $\frac{1 - z_0}{z_0} = e^{u_0 + v_0 \sqrt{-1}}$. Let λ and ξ be zero. Then we have

$$(2.4) \quad \varphi_{0,0}(\theta) = \frac{1}{(e^{-\theta} + 1)^2}$$

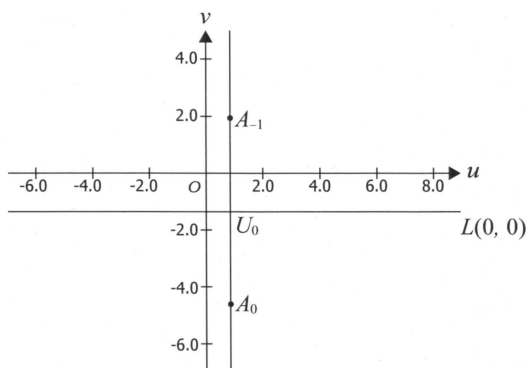


Fig 2.2: The line $L(0, 0)$

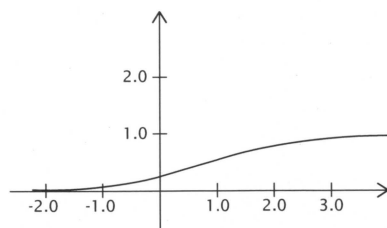


Fig 2.3: The generalized logistic curve on $L(0, 0)$

3 Graphs of $\varphi_{\lambda, \xi}(\theta)$

In this section, we fix $\alpha = 1.0 + 0.5\sqrt{-1}$ and $z_0 = 0.2 + 0.3\sqrt{-1}$. Then we get $U_0 = (0.86, -1.34)$. If we give λ and ξ , we remember the line $L(\lambda, \xi)$:

$$(3.1) \quad U = \theta \cdot \vec{g}_\lambda + U_0 + (0, \zeta)$$

and the function $\varphi_{\lambda, \zeta}(\theta)$ on $L(\lambda, \zeta)$:

$$(3.2) \quad \varphi_{\lambda, \zeta}(\theta) = \frac{1}{e^{-2\theta \frac{1}{\sqrt{1+\lambda^2}}} + 2e^{-\theta \frac{1}{\sqrt{1+\lambda^2}}} \cos\left(\theta \frac{\lambda}{\sqrt{1+\lambda^2}} + \zeta\right) + 1}$$

3.1 the level curves of $|\tilde{Z}(U)|^2$

To investigate the graph of $\varphi_{\lambda, \zeta}(\theta)$, we show the level curves of $|\tilde{Z}(U)|^2$ on $\mathbf{R}^2$

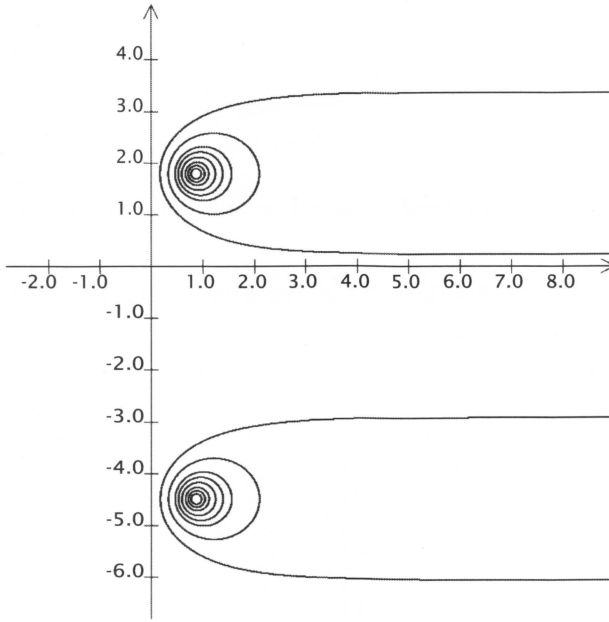


Fig 3.1: Level curves of $|\tilde{Z}(U)|^2$

3.2 A special case of $\lambda = 0$

In order to look for local maximal or minimal points of $\varphi_{0, \zeta}(\theta)$, we begin to calculate directly on the first derived function of $\varphi_{0, \zeta}(\theta)$

$$(3.3) \quad \frac{d\varphi_{0, \zeta}(\theta)}{d\theta} = \frac{2e^{-\theta}(e^{-\theta} + \cos \zeta)}{(e^{-2\theta} + 2e^{-\theta} \cos \zeta + 1)^2}$$

The solution of the equation $\frac{d\varphi_{0,\xi}(\theta)}{d\theta} = 0$, that is, $e^{-\theta} = -\cos \xi$ is local maximal or minimal points of $\varphi_{0,\xi}(\theta)$.

Summary of a special case

- (1) When ξ is $0 \leq \xi \leq \frac{\pi}{2}$, $\varphi_{0,\xi}(\theta)$ is the increase function such that $\lim_{\theta \rightarrow +\infty} \varphi_{0,\xi}(\theta) = 1$ and $\lim_{\theta \rightarrow -\infty} \varphi_{0,\xi}(\theta) = 0$.
- (2) When ξ is $\frac{\pi}{2} < \xi < \pi$, $\varphi_{0,\xi}(\theta)$ has one maximal point and two points of inflection such that $\lim_{\theta \rightarrow +\infty} \varphi_{0,\xi}(\theta) = 1$ and $\lim_{\theta \rightarrow -\infty} \varphi_{0,\xi}(\theta) = 0$.

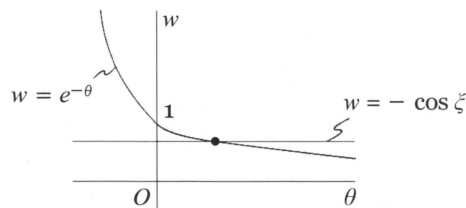


Fig 3.2: look for local maximal or minimal points of $\varphi_{0,\xi}(\theta)$

Moreover, we denote by $\theta(\xi)$ the point of the θ -axis which attains the maximal value of $\varphi_{0,\xi}(\theta)$. Then we get an inequation:

$$(3.4) \quad \theta(\bar{\xi}) > \theta(\underline{\xi}) \quad \text{if } \bar{\xi} < \underline{\xi}$$

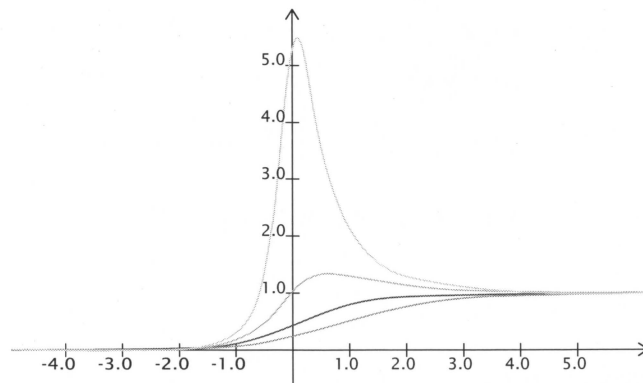


Fig 3.3: Some curves of $\varphi_{0,\xi}(\theta)$ (ξ is respectively 0, 1.4, 2.1, 2.7)

3.3 Another special case of $\xi = 0$

We also calculate on the first derived function of $\varphi_{\lambda,0}(\theta)$:

$$(3.5) \quad \frac{d\varphi_{\lambda,0}(\theta)}{d\theta} = \frac{2e^{-\theta \frac{1}{\sqrt{1+\lambda^2}}} \left\{ \frac{1}{\sqrt{1+\lambda^2}} e^{-\theta \frac{1}{\sqrt{1+\lambda^2}}} + \cos\left(\theta \frac{\lambda}{\sqrt{1+\lambda^2}} - \zeta\right) \right\}}{\left\{ e^{-2\theta \frac{1}{\sqrt{1+\lambda^2}}} + 2e^{-\theta \frac{1}{\sqrt{1+\lambda^2}}} \cos\left(\theta \frac{\lambda}{\sqrt{1+\lambda^2}}\right) + 1 \right\}^2}$$

where ζ is defined by $\cos \zeta = \frac{1}{\sqrt{1+\lambda^2}}$ and $\sin \zeta = \frac{\lambda}{\sqrt{1+\lambda^2}}$

Since we cannot solve explicitly the equation:

$$(3.6) \quad \frac{1}{\sqrt{1+\lambda^2}} e^{-\theta \frac{1}{\sqrt{1+\lambda^2}}} = -\cos\left(\theta \frac{\lambda}{\sqrt{1+\lambda^2}} - \zeta\right)$$

We try to investigate solutions of Equation (3.6) by computer approximation.

(i) When λ is sufficiently small, we may expect $\frac{1}{\sqrt{1+\lambda^2}} \doteq 1$, $\frac{\lambda}{\sqrt{1+\lambda^2}} \doteq 0$ and $\zeta \doteq 0$

If θ satisfies $\theta \frac{\lambda}{\sqrt{1+\lambda^2}} < \frac{\pi}{2}$ i.e. $\theta < \frac{\sqrt{1+\lambda^2}}{\lambda} \cdot \frac{\pi}{2}$, then the equation

$e^{-\theta} = -\cos\left(\theta \frac{\lambda}{\sqrt{1+\lambda^2}}\right)$ has no solution.

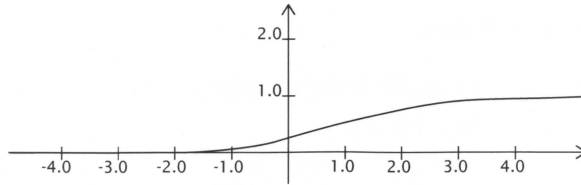


Fig 3.4: graph of $\varphi_{0,1,0}(\theta)$

We note that the two conditions: $\theta < \frac{\sqrt{1+\lambda^2}}{\lambda} \cdot \frac{\pi}{2}$ and λ is sufficiently small, mean that θ reaches a sufficiently large number.

(ii) When λ is sufficiently large, we may expect $\frac{1}{\sqrt{1+\lambda^2}} \doteq 0$, $\frac{\lambda}{\sqrt{1+\lambda^2}} \doteq 1$ and $\zeta \doteq \frac{\pi}{2}$.

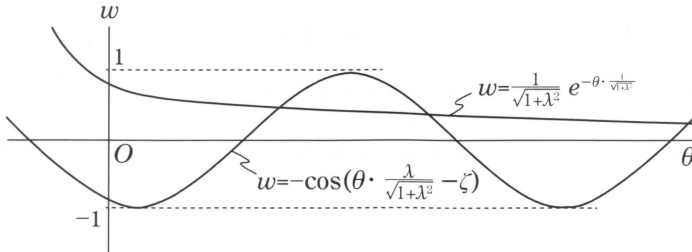


Fig 3.5: to solve Equation (3.6).

Since Equation (3.6) is $0 \cdot e^{-\theta \cdot 0} = 0 \doteq -\cos\left(\theta - \frac{\pi}{2}\right)$, we get the infinite solutions

$\theta \doteq \pi, 2\pi, 3\pi, \dots$. (See Fig 3.7)

Moreover, we remark that θ attains its local maximal points and minimal ones alternately.

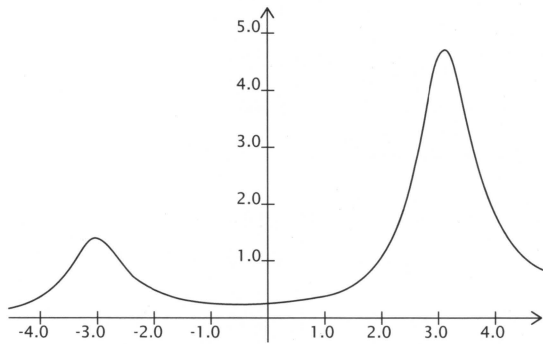


Fig 3.6: graph of $\varphi_{5,0}(\theta)$

Summary of another special case

- (1) If λ is sufficiently small, $\varphi_{\lambda,0}(\theta)$ is the increase function such that $\lim_{\theta \rightarrow +\infty} \varphi_{\lambda,0}(\theta) = 1$ and $\lim_{\theta \rightarrow -\infty} \varphi_{\lambda,0}(\theta) = 0$. (See Fig 3.4)
- (2) There exist λ such that $\varphi_{\lambda,0}(\theta)$ has many maximal and minimal points. (See Fig 3.5, Fig 3.6)

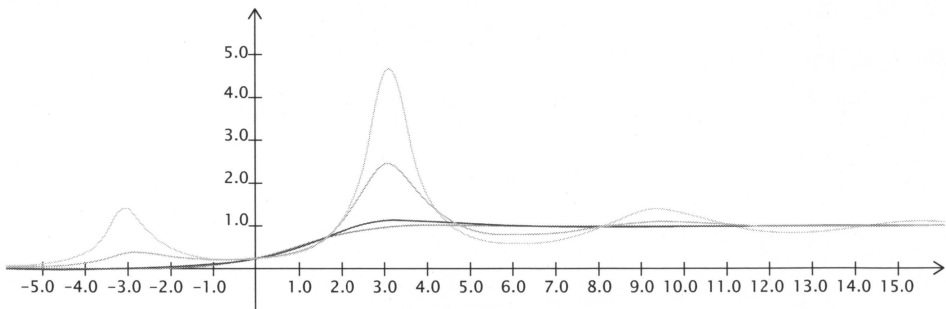


Fig 3.7: Some curves of $\varphi_{\lambda,0}(\theta)$ (λ is respectively 0.5, 1, 3, 5)

3.4 A generic case (i.e. $\lambda \neq 0$ and $\zeta \neq 0$)

As a generic case, let $\lambda = 3.0$ and $\zeta = 2.5$. (See Fig 3.8)

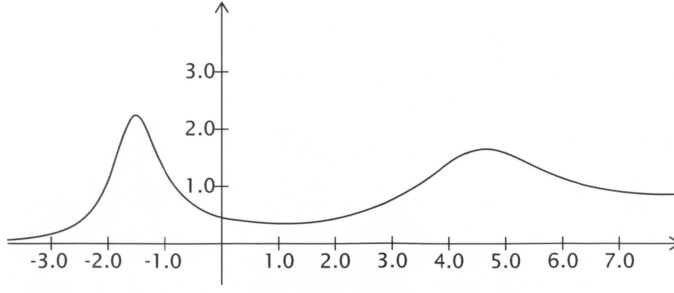


Fig 3.8: graph of $\varphi_{3,2.5}(\theta)$

4 Application to COVID-19

In accordance with the global spread of new-type coronavirus infections (COVID-19), WHO and health authorities in each country predicted the epidemic, and its application to infectious disease control has attracted great attention. Although the SEIR model, which is one of the typical predictive models, is based on a system of simple differential equations ($\frac{dx}{dt} = ax$, or $\frac{dx}{dt} = ax(1 - x)$ originated in the population dynamics by [2] and [3]), it has had considerable success in predicting the spread and end of the epidemic according to whether the prevention measures are adequate or not.

In the previous sections, we investigated properties of the generalized logistic curves $\varphi(\theta)$. Using these generalized logistic curves, we will, in this section, predict the variation of the number of new COVID-19 infections in Japan.

In order to approximate to the curve of Fig 4.1, we have the following three steps.

- Step1: We envisage level curves of $|\tilde{Z}(U)|^2$ on $\mathbf{R}^2$ (See Fig 3.1) and generalized logistic curves (See Fig 3.3, Fig 3.4, Fig 3.6, Fig 3.8) Especially, we give attention to pole points (See Fig 2.1).
- Step2: We will try to choose a line segment several times. Moreover, we illustrate the generalized logistic curve $\varphi(\theta)$ which is defined by the restriction $|\tilde{Z}(U)|^2$ on the line segment.
- Step3: Comparing these generalized logistic curves with the curve of Fig 4.1, we can fix one generalized logistic curve.

Example 4.1 *Fig 4.1 is the data of the variation of the number of new COVID-19 infections in Japan.*

We approximate Fig 4.1 by the generalized logistic curve $\varphi(\theta)$, (See Fig 4.2 and Fig 4.3.)

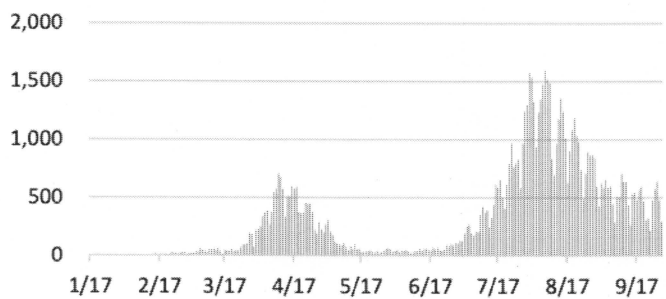


Fig 4.1: New COVID-19 infections in Japan (Ministry of Health Labour and Welfare)

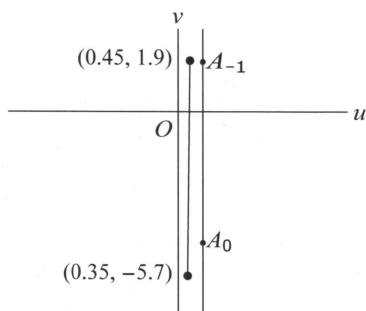


Fig 4.2: The line segment for new infections in Japan

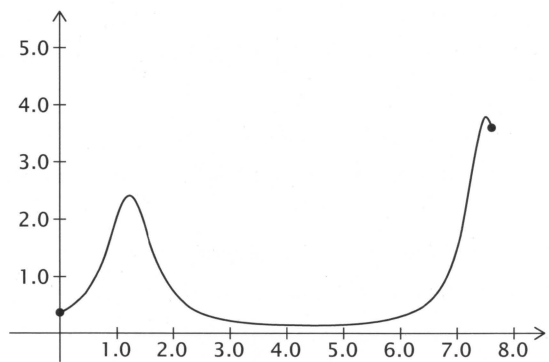


Fig 4.3: Generalized logistic curve for new infections in Japan

Example 4.2 *Fig 4.4 is the data of the variation of the cumulative number of COVID-19 infections in Japan.*

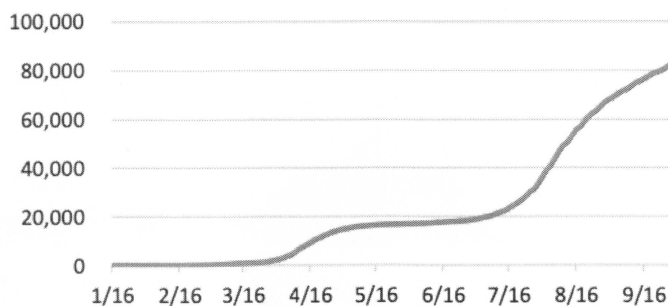


Fig 4.4: the cumulative number of COVID-19 infections in Japan (Ministry of Health Labour and Welfare)

We approximate Fig 4.4 by the generalized logistic curve $\varphi(\theta)$, (See Fig 4.5 and Fig 4.6.)

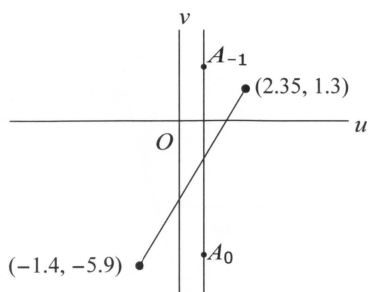


Fig 4.5: The line segment for the cumulative number of infections in Japan

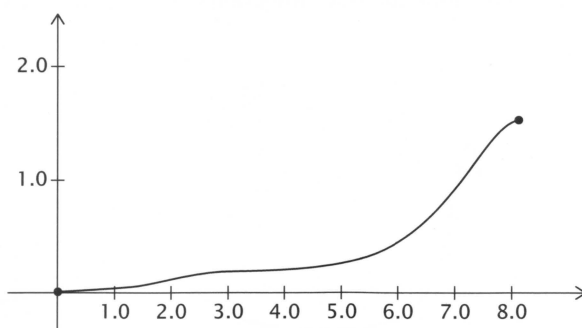


Fig 4.6: generalized logistic curve for the cumulative number of infections in Japan

Example 4.3 Fig 4.7 is the data of the variation of the number of new COVID-19 infections in the District of Columbia.

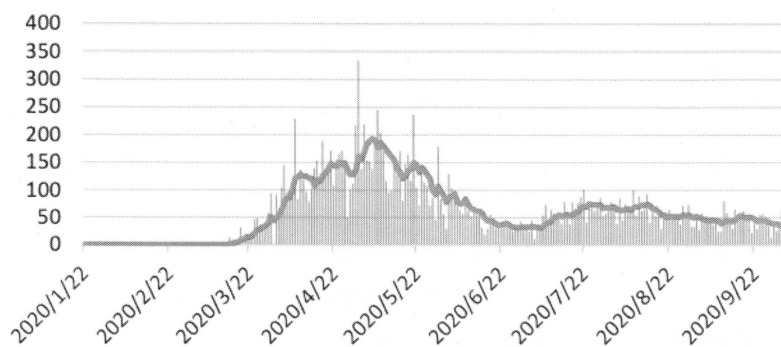


Fig 4.7: Daily Trends in Number of COVID-19 Cases in the District of Columbia Reported to CDC

We approximate Fig 4.7 by the generalized logistic curve $\varphi(\theta)$, (See Fig 4.8 and Fig 4.9.)

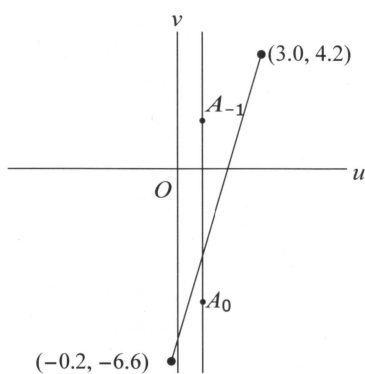


Fig 4.8: The line segment for new infections in the District of Columbia

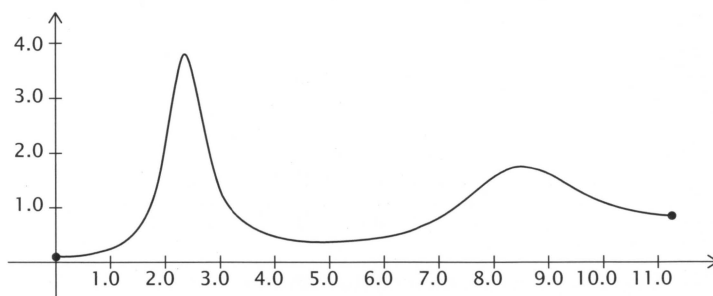


Fig 4.9: generalized logistic curve for new infections in the District of Columbia

Conclusion

We introduced the generalized logistic curve associated with the complex logistic equation.

We discovered many different properties of $\varphi_{i,\varsigma}(\theta)$. Using these properties, we could depict variations of the number of COVID-19 infections.

Acknowledgement

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