博 士 論 文

All Global Bifurcation Curves for a Cell Polarization Model

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Contents

1 Introduction

We are interested in wave-pinning in a reaction-diffusion model for cell polarization proposed by S.Ishihara, M.Otsuji and A.Mochizuki[2],M.Otsuji, et al.[12] and Y.Mori, A.Jilkine and L.Edelstein-Keshet[11].

The wave-pinning is such a phenomenon that a wave of activation of the species is initiated at one end of the domain, moves into the domain, decelerates, and eventually stops inside the domain, forming a stationary front.

We investigate a model proposed in [11]. The model is

$$
\text{(TP)} \begin{cases} \varepsilon W_t = \varepsilon^2 W_{xx} + W(W-1)(V+1-W) & \text{in } (0,1)\times(0,\infty), \\ \varepsilon V_t = \ D\ V_{xx} - W(W-1)(V+1-W) & \text{in } (0,1)\times(0,\infty), \\ W_x(0,t) = W_x(1,t) = 0, \quad V_x(0,t) = V_x(1,t) = 0 & \text{in } (0,\infty), \\ W(x,0) = W_0(x), \quad V(x,0) = V_0(x) & \text{in } (0,1), \end{cases}
$$

where $W = W(x,t)$ denotes the density of an active protein, $V = V(x,t)$ denotes the density of an inactive protein, ε and D are diffusion coefficients, $W_0(x)$ denotes the initial density of the active protein, and $V_0(x)$ denotes the initial density of the inactive protein.

It is easy to see that the mass conservation

$$
\int_0^1 (W(x,t)+V(x,t))dx = \int_0^1 (W_0(x)+V_0(x))dx = m
$$

holds, where m is the total mass determined by the mass of the initial densities $W_0(x)$ and $V_0(x)$.

Letting $D \to \infty$ in (TP), we formally obtain the following time dependent limiting equation:

$$
\text{(TLP)} \begin{cases} \varepsilon W_t = \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) & \text{in } (0,1)\times(0,\infty), \\ \varepsilon \dfrac{d\tilde{V}}{dt} = -\displaystyle\int_0^1 W(W-1)(\tilde{V}+1-W)dx & \text{in } (0,\infty), \\ W_x(0,t) = W_x(1,t) = 0 \quad \text{in } (0,\infty), \\ W(x,0) = W_0(x) \quad \text{in } (0,1), \quad \tilde{V}(0) = \tilde{V}_0, \end{cases}
$$

where $W = W(x,t)$, $\tilde{V} = \tilde{V}(t)$ is the density depending only on t. $W_0(x)$ denotes the initial density, and $\tilde{V}_0$ denotes an initial constant density.

Owing to the mass conservation, the stationary problem of (TP) can be reduced to the following Neumann problem with a nonlocal constraint:

$$(\mathrm{SP})\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(V+1-W) = 0 \quad \text{in } (0,1), \\ D\, V_{xx} - W(W-1)(V+1-W) = 0 \quad \text{in } (0,1), \\ W(x) > 0, \quad V(x) > 0 \quad \text{in } (0,1), \\ W_x(0) = W_x(1) = 0, \quad V_x(0) = V_x(1) = 0, \\ \displaystyle\int_0^1 (W(x) + V(x))dx = m, \end{cases}$$

where $W = W(x)$, $V = V(x)$, and m is a given initial total mass determined by initial densities.

Straight understanding of a stationary limiting problem for (TLP) is

$$\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0 \quad \text{in } (0,1), \\ \displaystyle\int_0^1 W(W-1)(\tilde{V}+1-W)dx = 0, \\ W_x(0) = W_x(1) = 0, \\ W(x) > 0 \quad \text{in } (0,1), \quad \tilde{V} > 0, \\ \displaystyle\int_0^1 W(x)dx + \tilde{V} = m. \end{cases}$$

The second equation automatically holds from the first and third equation. Hence the above system is equivalent to

$$\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0 \quad \text{in } (0,1), \\ W_x(0) = W_x(1) = 0, \\ W(x) > 0 \text{ in } (0,1), \quad \tilde{V} > 0, \\ \displaystyle\int_0^1 W(x)dx + \tilde{V} = m. \end{cases}$$

For simplicity we concentrate on monotone increasing solutions, since we can obtain other solutions by reflecting this kind of solutions. Thus, we get

$$\begin{cases} \varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0 \quad \text{in } (0,1), \\ W_x(0) = W_x(1) = 0, \\ W(0) > 0, \quad W_x(x) > 0 \text{ in } (0,1), \quad \tilde{V} > 0, \\ \displaystyle\int_0^1 W(x)dx + \tilde{V} = m. \end{cases}$$

Here it should be noted that we may omit the condition $W(0) > 0$, since this condition follows from other conditions. Thus we obtain a stationary limiting problem as

$$
\text{(SLP)}\begin{cases}
\varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0 \quad \text{in } (0,1), & (1.1)\\
W_x(0) = W_x(1) = 0, & (1.2)\\
W_x(x) > 0 \text{ in } (0,1), \quad \tilde{V} > 0, & (1.3)\\
\displaystyle\int_0^1 W(x)dx + \tilde{V} = m, & (1.4)
\end{cases}
$$

where m and ε are given positive constants, $W = W(x)$ is an unknown function, and $\tilde{V}$ is an unknown nonnegative constant.

Interesting bifurcation diagrams are obtained in [11] by numerical computations. Kuto and Tsujikawa [6] obtained several mathematical results for (SLP) with suitable change of variables(see, also [4] and [5]). We have obtained the exact expressions of all the solutions for it by using the Jacobi elliptic functions and complete elliptic integrals in Mori, Kuto, Nagayama, Tsujikawa and Yotsutani [8]. The method to obtain all the exact solutions essentially based on the method which started in Lou, Ni and Yotsutani [7]. It is developed by Kosugi, Morita and Yotsutani [3] to investigate the Cahn-Hilliard equation treated in Carr, Gurtin and Semrod [1].

Now, let us introduce an auxiliary problem to investigate (SLP). Let $\tilde{V} > 0$ be given, let us consider the problem

$$
(\mathrm{AP};\tilde{\mathrm{V}})\begin{cases}
\varepsilon^2 W_{xx} + W(W-1)(\tilde{V}+1-W) = 0 \quad \text{in } (0,1), & (1.5)\\
W_x(0) = W_x(1) = 0, & (1.6)\\
W_x(x) > 0 \text{ in } (0,1). & (1.7)
\end{cases}
$$

The following fact is fundamental (see, e.g. Smoller and Wasserman [14], Smoller [13], and Theorem 2.1 in [8]).

There exists a solution of (AP;$\tilde{V}$), if and only if $(\tilde{V},\ \varepsilon^2) \in \mathcal{G}$, where

$$
\mathcal{G} := \left\{ (\tilde{V},\ \varepsilon^2) :\ 0 < \varepsilon^2 < \frac{\tilde{V}}{\pi^2} \right\}. \tag{1.8}
$$

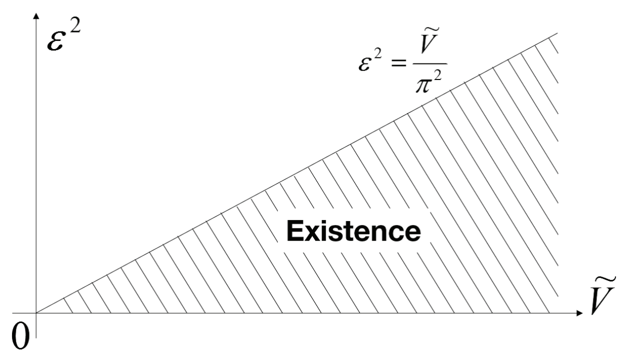

Figure 1.1: Existence region of solution for (AP;$\tilde{V}$)

Moreover, the solution is unique, is represented by elliptic integrals, and has properties

$$0 < W(x; \tilde{V}, \varepsilon^2) < \tilde{V} + 1, \tag{1.9}$$

$$W(x; \tilde{V}, \varepsilon^2) = \tilde{V} + 1 - \tilde{V} \cdot W\left(1 - x; \frac{1}{\tilde{V}}, \frac{\varepsilon^2}{\tilde{V}^2}\right). \tag{1.10}$$

Let us define the global bifurcation sheet S by

$$S := \left\{ \left(\tilde{V}, \varepsilon^2, \boldsymbol{m}(\tilde{V},\ \varepsilon^2) \right) : (\tilde{V},\ \varepsilon^2) \in \mathcal{G} \right\}, \tag{1.11}$$

where

$$\boldsymbol{m}(\tilde{V},\ \varepsilon^2) := \int_0^1 W(x; \tilde{V}, \varepsilon^2) dx + \tilde{V}. \tag{1.12}$$

We note that

$$\boldsymbol{m}(\tilde{V}, \varepsilon^2) = 2\tilde{V} + 2 - \tilde{V}\ \boldsymbol{m}\left(\frac{1}{\tilde{V}}, \frac{\varepsilon^2}{\tilde{V}^2}\right) \quad \text{for any} \quad \tilde{V} > 0, \quad \varepsilon > 0 \tag{1.13}$$

by (1.10), and which implies

$$\boldsymbol{m}(1, \varepsilon^2) = 2 \quad \text{for any} \quad \varepsilon^2 \in \left(0, \frac{1}{\pi^2}\right). \tag{1.14}$$

We see from Theorem 4.2 that $\boldsymbol{m}(\tilde{V},\ \varepsilon^2)$ is represented by complete elliptic integrals. For given $m > 0$, level curve with the height m of the global bifurcation sheet S corresponds to the bifurcation diagram in the plane $(\tilde{V}, \varepsilon^2)$ for (SLP) with given m. Thus, for each m, we can obtain the bifurcation diagram by

$$\left\{ (\tilde{V},\ \varepsilon^2) \in \mathcal{G} : \boldsymbol{m}(\tilde{V},\ \varepsilon^2) = m \right\}. \tag{1.15}$$

In Figure 1.2, we show the global bifurcation sheet and bifurcation diagrams of (SLP) which is obtained numerically in Section 9. In this paper we prove the following Theorems by obtaining representation formula for the global bifurcation sheet and analyzing it.

Theorem 1.1 *Let $0 < m \leq 1$ be given. There exists no solution of (SLP).*

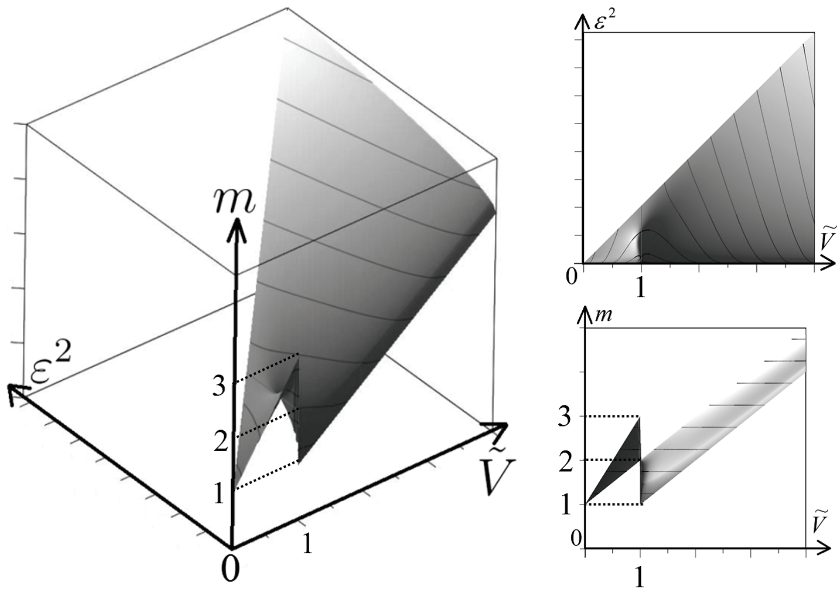

Figure 1.2: Global bifurcation sheet for (SLP)

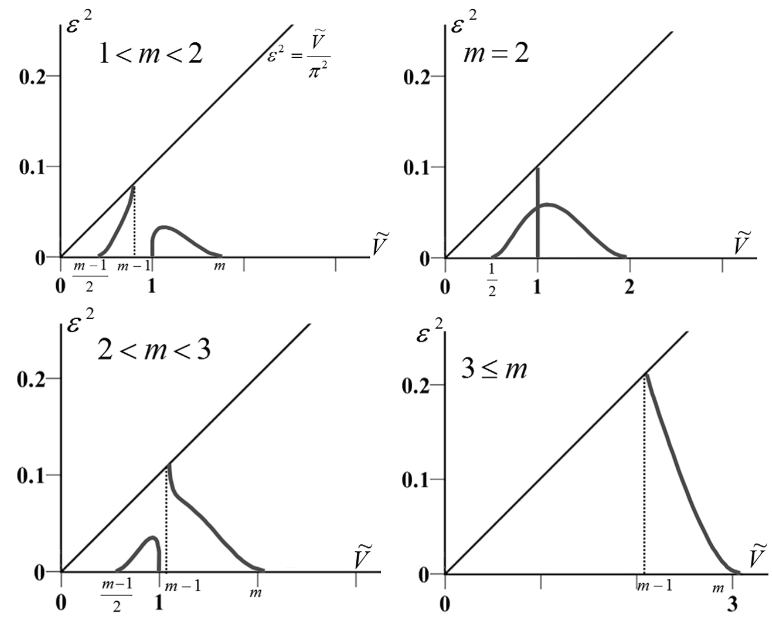

Figure 1.3: Bifurcation diagrams for each m

Theorem 1.2 *Let $1 < m < 2$ be given. The following holds.*

(i) *There exists no solution of (SLP) for $\tilde{V} \in (0,\ (m-1)/2] \cup [m-1,\ 1] \cup [m, \infty)$.*

(ii) *There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP) for $\tilde{V} \in ((m-1)/2,\ m-1)$. Moreover, $\varepsilon(\tilde{V})$ is continuous on $[(m-1)/2,\ m-1]$ by defining $\varepsilon((m-1)/2) = 0$, $\varepsilon(m-1) = \sqrt{m-1}/\pi$.*

(iii) *There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP) for $\tilde{V} \in (1,\ m)$. Moreover, $\varepsilon(\tilde{V})$ is continuous on $[1,\ m]$, by defining $\varepsilon(1) = 0$, $\varepsilon(m) = 0$.*

We show several profiles of $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ corresponding to $(\tilde{V}, \varepsilon^2(\tilde{V}))$ assured by Theorem 1.2 in Figure 1.4.

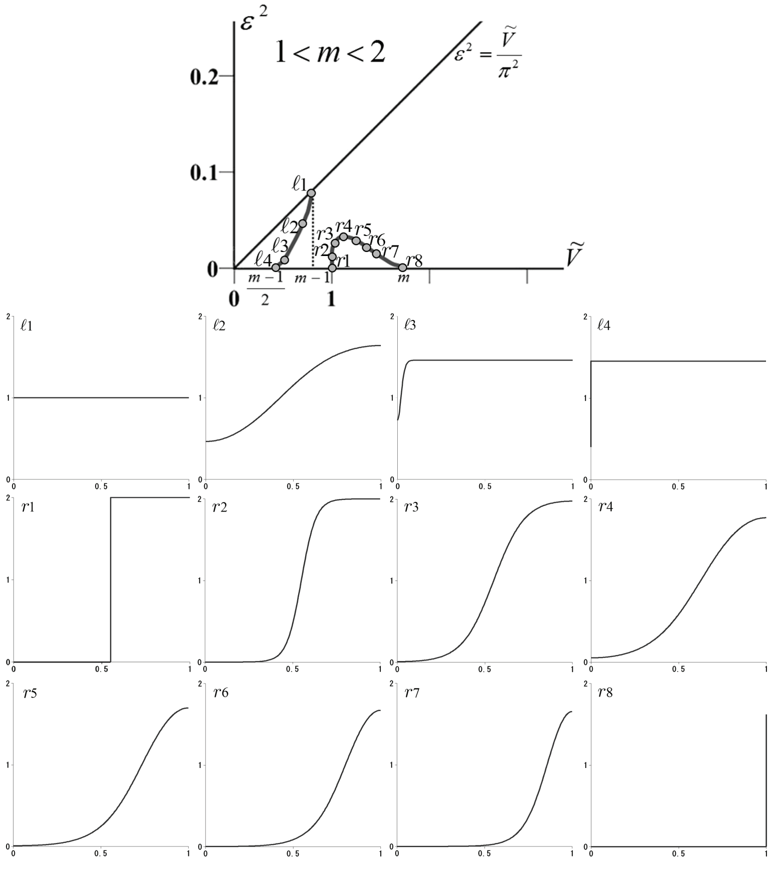

Figure 1.4: Profiles of $W(x; \tilde{V}, \varepsilon^2)$ for $1 < m < 2$.

Theorem 1.3 *Let $m = 2$ be given. The following holds.*

(i) *There exists no solution of (SLP) for $\tilde{V} \in (0,\ 1/2] \cup [2, \infty)$.*

(ii) *There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP) for $\tilde{V} \in (1/2,\ 1)$. Moreover, there exists the unique $\varepsilon_* = 0.23529\cdots$ such that $\varepsilon(\tilde{V})$ is continuous on $[1/2,\ 1]$ by defining $\varepsilon\,(1/2) = 0$, $\varepsilon(1) = \varepsilon_*$.*

(iii) *For $\tilde{V} = 1$, there exists no solution of (SLP) for $\varepsilon \in [1/\pi, \infty)$, and there exists the unique solution $W(x; 1, \varepsilon^2)$ of (SLP) for $\varepsilon \in (0, 1/\pi)$.*

(iv) *There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP) for $\tilde{V} \in (1,\ 2)$. Moreover, $\varepsilon(\tilde{V})$ is continuous on $[1,\ 2]$ by defining $\varepsilon(1) = \varepsilon_*$, $\varepsilon\,(2) = 0$.*

We show several profiles of $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ corresponding to $(\tilde{V}, \varepsilon^2(\tilde{V}))$ assured by Theorem 1.3 in Figure 1.5.

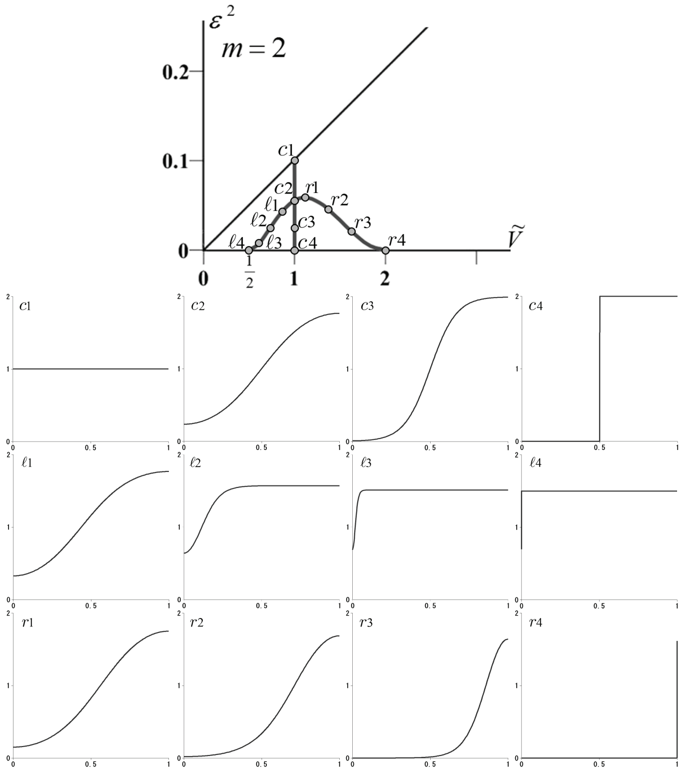

Figure 1.5: Profiles of $W(x; \tilde{V}, \varepsilon^2)$ for $m = 2$.

Theorem 1.4 *Let $2 < m < 3$ be given. The following holds.*

(i) There exists no solution of (SLP) for $\tilde{V} \in (0,\ (m-1)/2] \cup [1,\ m-1] \cup [m, \infty)$.

(ii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP) for $\tilde{V} \in ((m-1)/2,\ 1)$. Moreover, $\varepsilon(\tilde{V})$ is continuous on $[(m-1)/2,\ 1]$ by defining $\varepsilon\left((m-1)/2\right) = 0,\ \varepsilon\left(1\right) = 0$.

(iii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP) for $\tilde{V} \in (m-1,\ m)$. Moreover, $\varepsilon(\tilde{V})$ is continuous on $[m-1,\ m]$ by defining $\varepsilon\left(m-1\right) = \sqrt{m-1}/\pi,\ \ \varepsilon\left(m\right) = 0$.

We show several profiles of $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ corresponding to $(\tilde{V}, \varepsilon^2(\tilde{V}))$ assured by Theorem 1.4 in Figure 1.6.

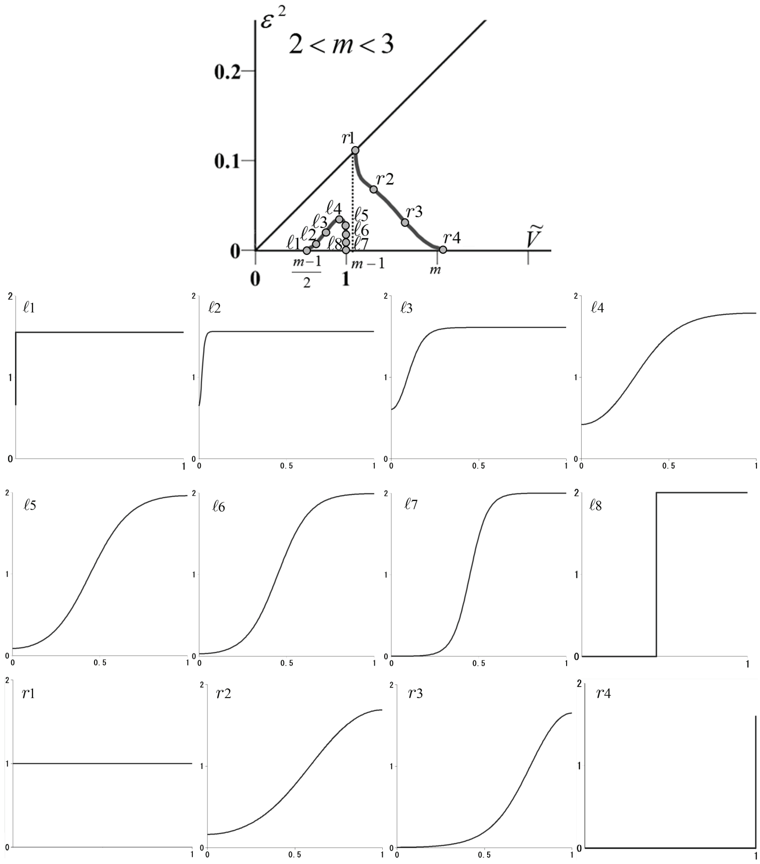

Figure 1.6: Profiles of $W(x; \tilde{V}, \varepsilon^2)$ for $2 < m < 3$.

Theorem 1.5 *Let $m \geq 3$ be given. The following holds.*

(i) *There exists no solution of (SLP) for $\tilde{V} \in (0, m-1] \cup [m, \infty)$.*

(ii) *There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ is a solution of (SLP) for $\tilde{V} \in (m-1,\ m)$. Moreover, $\varepsilon(\tilde{V})$ is continuous on $[m-1,\ m]$ by defining $\varepsilon\,(m-1) = \sqrt{m-1}/\pi,\ \ \varepsilon\,(m) = 0$.*

We show several profiles of $W(x; \tilde{V}, \varepsilon^2(\tilde{V}))$ corresponding to $(\tilde{V}, \varepsilon^2(\tilde{V}))$ assured by Theorem 1.5 in Figure 1.7.

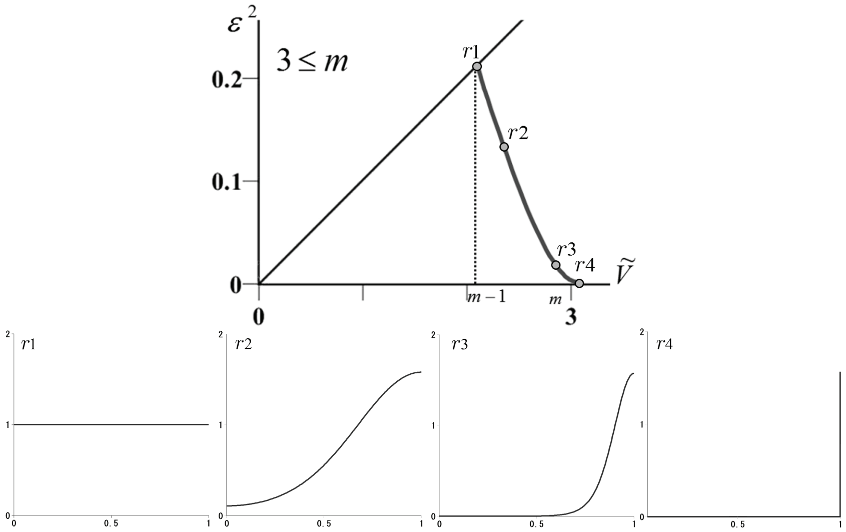

Figure 1.7: Profiles of $W(x; \tilde{V}, \varepsilon^2)$ for $m \geq 3$.

This paper is organized as follows.

In Section 2 we give the definition for the elliptic functions and the complete elliptic integrals as a preliminary. In Section 3 we state Proposition 3.1 and Theorems 3.1-3.10 which are used for the proofs of main theorems, and give proofs of Theorems 1.1-1.5. Proposition 3.1 gives limits of $\boldsymbol{m}(\tilde{V}, \varepsilon^2)$ as $\varepsilon^2 \to 0$ and $\tilde{V}/\pi^2$. Theorem 3.1 gives monotonicity of $\boldsymbol{m}(\tilde{V}, \varepsilon^2)$ in ε. In Section 4 we show Theorems 4.1 and 4.2, which represent all exact solutions for $(AP; \tilde{V})$ and an expression of $\boldsymbol{m}(\tilde{V}, \varepsilon^2)$. In Section 5 we show a proof of Proposition 3.1. In Section 6 we give a proof of Theorem 3.1 by using Propositions 6.1-6.6. Propositions 6.1 and 6.2 give the expression of $\partial \boldsymbol{m}(\tilde{V}, \varepsilon^2)/\partial \varepsilon$ by using parameters h and s. Propositions 6.3 and 6.4 give properties of $\mathcal{J}(h, s)$ whose positivity gives negativity of $\partial \boldsymbol{m}(\tilde{V}, \varepsilon^2)/\partial \varepsilon$. Propositions 6.5 and 6.6 give that $\partial \mathcal{J}(h, s)/\partial s$ has the unique zero. In Section 7 we give a proof of Proposition 6.6. In Section 8 we explain the existence and uniqueness of secondary bifurcation point, which is stated in Theorem 3.8. In

Section 9 we explain numerical results about the stability of solutions of (SLP). Finally, we show conclusion remarks in Section 10.

2 Preliminary

In this section, we give the definition for the elliptic functions and the complete elliptic integrals which are used in this paper. Let $\mathrm{sn}(x,k)$ and $\mathrm{cn}(x,k)$ be Jacobi's elliptic functions. The following properties holds:

$$\mathrm{sn}^{-1}(z,k) = \int_0^z \frac{1}{\sqrt{1-k^2\xi^2}\sqrt{1-\xi^2}}\,d\xi \quad (-1 \le z \le 1,\ 0 < k < 1),$$

$$\mathrm{sn}^2(x,k) + \mathrm{cn}^2(x,k) = 1, \quad \mathrm{cn}(0,k) = 1.$$

Let $k \in [0,1)$ and $-1 < \nu < 1$. The complete elliptic integrals of the first, second and third kind are defined by

$$K(k) := \int_0^1 \frac{1}{\sqrt{1-k^2t^2}\sqrt{1-t^2}}\,dt, \quad E(k) := \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\,dt,$$

and

$$\Pi(\nu,k) := \int_0^1 \frac{1}{(1+\nu t^2)\sqrt{1-k^2t^2}\sqrt{1-t^2}}\,dt,$$

respectively. We see that $K(k)$ is monotone increasing in k,

$$K(0) = \frac{\pi}{2}, \quad \lim_{k\to 1} K(k) = \infty$$

and $E(k)$ is monotone decreasing in k,

$$E(0) = \frac{\pi}{2}, \quad \lim_{k\to 1} E(k) = 1.$$

We show graphs of complete elliptic integrals.

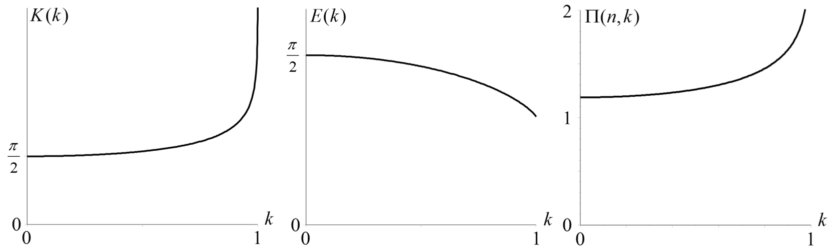

Figure 2.1: complete elliptic integrals $K(k)$, $E(k)$ and $\Pi(3/4,k)$.

The following formulas for the complete elliptic integrals are fundamental:

$$\frac{d}{dk}K(k) = \frac{E(k)}{(1-k^2)k} - \frac{K(k)}{k}. \quad \frac{d}{dk}E(k) = \frac{E(k)}{k} - \frac{K(k)}{k}.$$

$$\frac{\partial}{\partial k}\Pi(\nu,k) = \frac{kE(k)}{(k^2+\nu)(1-k^2)} - \frac{k\Pi(\nu,k)}{k^2+\nu}.$$

$$\frac{\partial}{\partial \nu}\Pi(\nu,k) = \frac{(k^2-\nu^2)\Pi(\nu,k)}{2(1+\nu)(k^2+\nu)\nu} - \frac{K(k)}{2(1+\nu)\nu} + \frac{E(k)}{2(1+\nu)(k^2+\nu)}.$$

$$\frac{d}{dh}K(\sqrt{h}) = \frac{E(\sqrt{h}) - (1-h)K(\sqrt{h})}{2h(1-h)}, \quad \frac{d}{dh}E(\sqrt{h}) = \frac{E(\sqrt{h}) - K(\sqrt{h})}{2h}.$$

$$\frac{d}{dH}K(\sqrt{1-H^2}) = \frac{E(\sqrt{1-H^2}) - H^2K(\sqrt{1-H^2})}{H(1-H^2)}.$$

$$\frac{d}{dH}E(\sqrt{1-H^2}) = \frac{H(K(\sqrt{1-H^2}) - E(\sqrt{1-H^2}))}{1-H^2}.$$

It is easy to see that the following inequalities hold.

Lemma 2.1 *It hold that*

$$\sqrt{1-h} < \frac{E(\sqrt{h})}{K(\sqrt{h})} < 1 - \frac{h}{2} < 1 \quad (0 < h < 1), \tag{2.1}$$

and

$$H < \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})} < \frac{1+H^2}{2} < 1 \quad (0 < H < 1). \tag{2.2}$$

We show graphs of inequalities (2.1) and (2.2).

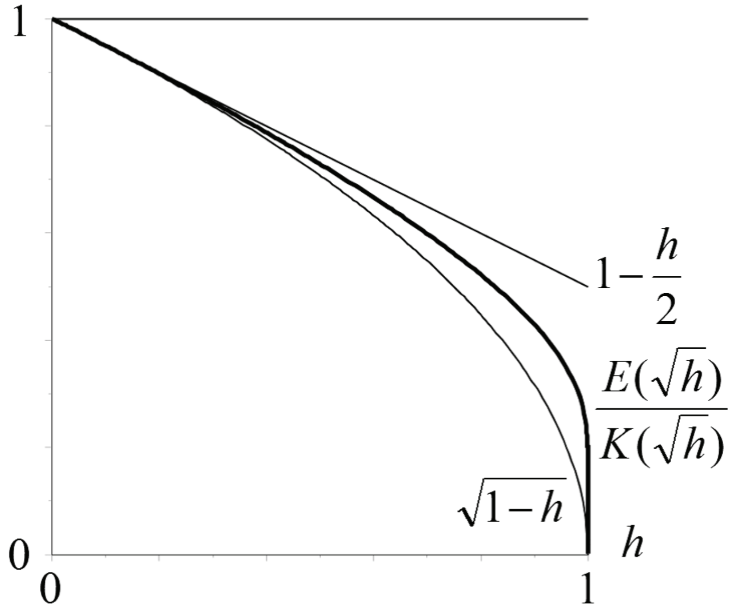

Figure 2.2: Profiles of inequalities (2.1).

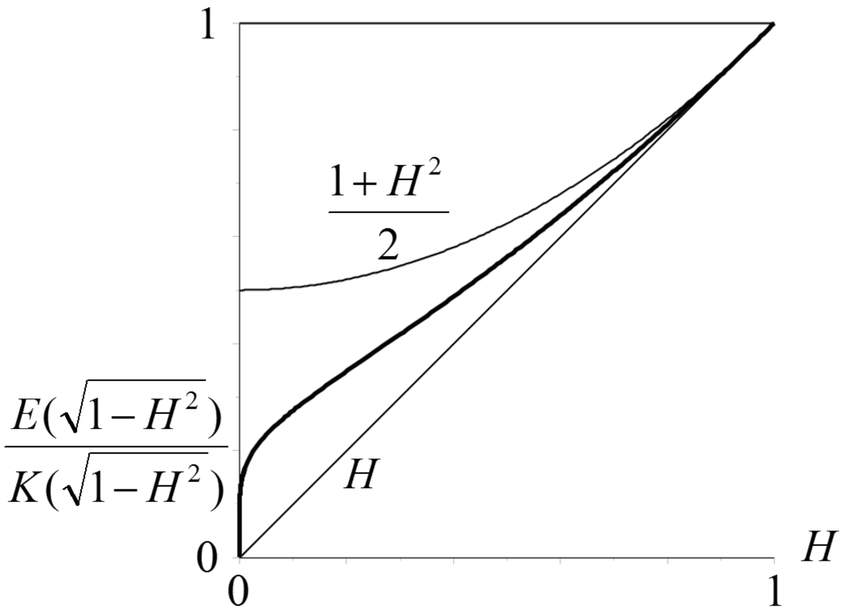

Figure 2.3: Profiles of inequalities (2.2).

3 Key theorems and proofs of main theorems

In this section we show crucial results of the paper which are basis of Theorems 1.1-1.5.

3.1 Properties of global bifurcation sheet

The function $\boldsymbol{m}(\tilde{V}, \varepsilon^2)$ defined by (1.12) has the following properties.

Proposition 3.1 *Let $\tilde{V} \in (0, \infty)$ be fixed. The following holds:*

$$(i)\quad \boldsymbol{m}(\tilde{V}, \varepsilon^2) \to \tilde{V} + 1 \quad \text{as} \quad \varepsilon^2 \to \tilde{V}/\pi^2. \tag{3.1}$$

$$(ii)\quad \text{For} \quad \tilde{V} \in (0, 1), \quad \boldsymbol{m}(\tilde{V}, \varepsilon^2) \to 2\tilde{V} + 1 \quad \text{as} \quad \varepsilon^2 \to 0. \tag{3.2}$$

$$(iii)\quad \text{For} \quad \tilde{V} \in (1, \infty), \quad \boldsymbol{m}(\tilde{V}, \varepsilon^2) \to \tilde{V} \quad \text{as} \quad \varepsilon^2 \to 0. \tag{3.3}$$

The following theorems are the most crucial results of the paper.

Theorem 3.1 *Let $\boldsymbol{m}(\tilde{V},\ \varepsilon^2)$ be the function defined by (1.12), and $\tilde{V} > 0$ be fixed. It holds that*

$$\frac{\partial \boldsymbol{m}(\tilde{V},\ \varepsilon^2)}{\partial \varepsilon} < 0 \quad \text{for} \quad \varepsilon^2 \in \left(0, \frac{\tilde{V}}{\pi^2}\right) \quad \text{with} \quad \tilde{V} \in (0, 1), \tag{3.4}$$

$$\frac{\partial \boldsymbol{m}(1,\ \varepsilon^2)}{\partial \varepsilon} \equiv 0 \quad \text{for} \quad \varepsilon^2 \in \left(0, \frac{1}{\pi^2}\right), \tag{3.5}$$

$$\frac{\partial \boldsymbol{m}(\tilde{V},\ \varepsilon^2)}{\partial \varepsilon} > 0 \quad \text{for} \quad \varepsilon^2 \in \left(0, \frac{\tilde{V}}{\pi^2}\right) \quad \text{with} \quad \tilde{V} \in (1, \infty). \tag{3.6}$$

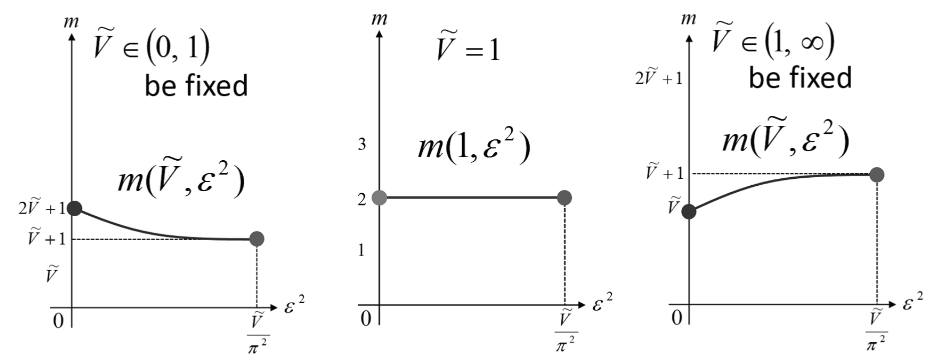

Figure 3.1: Graph of $\partial \boldsymbol{m}(\tilde{V}, \varepsilon^2)/\partial \varepsilon$.

As applications of Proposition 3.1 and Theorem 3.1 which are prove in Section 5 and 6, respectively. we obtain the following Theorems.

Theorem 3.2 *Let $0 < m \leq 1$ be given. It holds that*

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : \tilde{V} > 0,\ 0 < \varepsilon^2 < \frac{\tilde{V}}{\pi^2} \right\} = (1, \infty). \tag{3.7}$$

Theorem 3.3 *Let $1 < m < 2$ be given. The following holds:*

(i) It holds that

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : 0 < \tilde{V} < \frac{m-1}{2} \right\} = (1, m), \tag{3.8}$$

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : m - 1 < \tilde{V} < 1 \right\} = (m, 2), \tag{3.9}$$

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : \tilde{V} > m \right\} = (m, \infty). \tag{3.10}$$

(ii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $\boldsymbol{m}(\tilde{V}, \varepsilon^2(\tilde{V})) = m$ for $\tilde{V} \in ((m-1)/2,\ m-1)$.

(iii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $\boldsymbol{m}(\tilde{V}, \varepsilon^2(\tilde{V})) = m$ for $\tilde{V} \in (1,\ m)$.

Theorem 3.4 *Let $m = 2$ be given. The following holds:*

(i) It holds that

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : 0 < \tilde{V} < \frac{1}{2} \right\} = (1, 2), \tag{3.11}$$

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : \tilde{V} > 2 \right\} = (2, \infty). \tag{3.12}$$

(ii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $\boldsymbol{m}(\tilde{V}, \varepsilon^2(\tilde{V})) = m$ for $\tilde{V} \in (1/2,\ 1)$.

(iii) For $\tilde{V} = 1$, there exists no solution of (SLP) for $\varepsilon \in [1/\pi, \infty)$, and there exists the unique solution $W(x; 1, \varepsilon^2)$ of (SLP) for $\varepsilon \in (0, 1/\pi)$.

(iv) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $\boldsymbol{m}(\tilde{V}, \varepsilon^2(\tilde{V})) = m$ for $\tilde{V} \in (1,\ 2)$.

Theorem 3.5 *Let $2 < m < 3$ be given. The following holds:*

(i) It holds that

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : 0 < \tilde{V} < \frac{m-1}{2} \right\} = (1, m), \tag{3.13}$$

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : 1 < \tilde{V} < m-1 \right\} = (1, m), \tag{3.14}$$

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : \tilde{V} > m \right\} = (m, \infty). \tag{3.15}$$

(ii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $\boldsymbol{m}(\tilde{V}, \varepsilon^2(\tilde{V})) = m$ for $\tilde{V} \in ((m-1)/2,\ 1)$.

(iii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $\boldsymbol{m}(\tilde{V}, \varepsilon^2(\tilde{V})) = m$ for $\tilde{V} \in (m-1,\ m)$.

Theorem 3.6 *Let $m \geq 3$ be given. The following holds:*

(i) It holds that

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : 0 < \tilde{V} < m-1 \right\} = (1, m), \tag{3.16}$$

$$\left\{ \boldsymbol{m}(\tilde{V}, \varepsilon^2) : \tilde{V} > m \right\} = (m, \infty). \tag{3.17}$$

(ii) There exists the unique $\varepsilon(\tilde{V}) \in (0, \sqrt{\tilde{V}}/\pi)$ such that $\boldsymbol{m}(\tilde{V}, \varepsilon^2(\tilde{V})) = m$ for $\tilde{V} \in (m-1,\ m)$.

Theorem 3.7 *Let $\varepsilon(\tilde{V})$ be appeared in (ii) and (iii) of Theorem 3.3. It has the following properties.*

$$\varepsilon(\tilde{V}) \longrightarrow 0 \quad as \quad \tilde{V} \downarrow \frac{m-1}{2}, \tag{3.18}$$

$$\varepsilon(\tilde{V}) \longrightarrow \frac{\tilde{V}}{\pi^2} \quad as \quad \tilde{V} \uparrow m-1, \tag{3.19}$$

$$\varepsilon(\tilde{V}) \longrightarrow 0 \quad as \quad \tilde{V} \downarrow 1, \tag{3.20}$$

$$\varepsilon(\tilde{V}) \longrightarrow 0 \quad as \quad \tilde{V} \uparrow m. \tag{3.21}$$

Theorem 3.8 *Let $\varepsilon(\tilde{V})$ be appeared in (ii) and (iv) of Theorem 3.4. It has the following properties.*

$$\varepsilon(\tilde{V}) \longrightarrow 0 \quad as \quad \tilde{V} \downarrow \frac{1}{2}, \tag{3.22}$$

$$\varepsilon(\tilde{V}) \longrightarrow \varepsilon_* \quad as \quad \tilde{V} \uparrow 1, \tag{3.23}$$

$$\varepsilon(\tilde{V}) \longrightarrow \varepsilon_* \quad as \quad \tilde{V} \downarrow 1, \tag{3.24}$$

$$\varepsilon(\tilde{V}) \longrightarrow 0 \quad as \quad \tilde{V} \uparrow 2, \tag{3.25}$$

where

$$\varepsilon_* := \frac{1}{K(\sqrt{h_*})\sqrt{2(2-h_*)}} = 0.23529\cdots,$$

and $h_* = 0.95285\cdots$ *is the unique solution of*

$$\begin{cases} \dfrac{2-h}{1-h}\cdot\dfrac{E(\sqrt{h})}{K(\sqrt{h})} - 8 = 0, \\ 0 < h < 1. \end{cases} \tag{3.26}$$

Here, $\mathcal{E}(h,s)$ *is defined by (4.7),* $K(\cdot)$ *and* $E(\cdot)$ *are the complete elliptic integrals of the first and second kind, respectively.*

Theorem 3.9 *Let* $\varepsilon(\tilde{V})$ *be appeared in (ii) and (iii) of Theorem 3.5. It has the following properties.*

$$\varepsilon(\tilde{V}) \longrightarrow 0 \quad as \quad \tilde{V} \downarrow \frac{m-1}{2}, \tag{3.27}$$

$$\varepsilon(\tilde{V}) \longrightarrow 0 \quad as \quad \tilde{V} \uparrow 1, \tag{3.28}$$

$$\varepsilon(\tilde{V}) \longrightarrow \frac{\tilde{V}}{\pi^2} \quad as \quad \tilde{V} \downarrow m-1, \tag{3.29}$$

$$\varepsilon(\tilde{V}) \longrightarrow 0 \quad as \quad \tilde{V} \uparrow m. \tag{3.30}$$

Theorem 3.10 *Let* $\varepsilon(\tilde{V})$ *be appeared in (ii) of Theorem 3.6. It has the following properties.*

$$\varepsilon(\tilde{V}) \longrightarrow \frac{\tilde{V}}{\pi^2} \quad as \quad \tilde{V} \downarrow m-1, \tag{3.31}$$

$$\varepsilon(\tilde{V}) \longrightarrow 0 \quad as \quad \tilde{V} \uparrow m. \tag{3.32}$$

3.2 Proofs of Theorems 1.1-1.5, 3.2-3.6, 3.7, 3.9-3.10

We now gives proofs of Theorems 1.1-1.5, 3.2-3.6, 3.7, 3.9 and 3.10.

Proofs of Theorems 1.1-1.5. Theorem 1.1 follows from Theorem 3.2. Theorem 1.2 follows from Theorem 3.3 and 3.7. Theorem 1.3 follows from Theorem 3.4 and 3.8. Theorem 1.4 follows from Theorem 3.5 and 3.9. Theorem 1.5 follows from Theorem 3.6 and 3.10.□

We give proofs of Theorems 3.2-3.6 by using Proposition 3.1 and Theorem 3.1 which we prove later. It is easy to see that the following lemmas hold.

Lemma 3.1 *Let $\tilde{V} > 0$ and $m > 0$. The following equivalence holds:*

(i) It holds that

$$\begin{cases} \tilde{V} + 1 < m < 2\tilde{V} + 1, \\ 0 < \tilde{V} < 1, \end{cases} \Leftrightarrow \begin{cases} \dfrac{m-1}{2} < \tilde{V} < m - 1, \\ 0 < \tilde{V} < 1, \end{cases}$$
$$\Leftrightarrow \max\left(0, \frac{m-1}{2}\right) < \tilde{V} < \min(1, m-1).$$

(ii) It holds that

$$\begin{cases} \tilde{V} < m < \tilde{V} + 1, \\ \tilde{V} > 1, \end{cases} \Leftrightarrow \begin{cases} m - 1 < \tilde{V} < m, \\ \tilde{V} > 1, \end{cases}$$
$$\Leftrightarrow \max(1, m-1) < \tilde{V} < m.$$

Lemma 3.2 *Let $\mathcal{D}_m$ be defined by*

$$\mathcal{D}_m := \left(\max\left(0, \frac{m-1}{2}\right), \min(1, m-1)\right) \cup \left(\max(1, m-1), m\right).$$

Then it holds that

$$\begin{aligned} &\mathcal{D}_m = \emptyset \quad \text{for} \quad 0 \le m \le 1, \\ &\mathcal{D}_m = \left(\frac{m-1}{2}, m-1\right) \cup (1, m) \quad \text{for} \quad 1 < m < 2, \\ &\mathcal{D}_m = \left(\frac{1}{2}, 1\right) \cup (1, 2) \quad \text{for} \quad m = 2, \\ &\mathcal{D}_m = \left(\frac{m-1}{2}, 1\right) \cup (m-1, m) \quad \text{for} \quad 2 < m < 3, \\ &\mathcal{D}_m = (m-1, m) \quad \text{for} \quad m \ge 3. \end{aligned}$$

Proofs of Theorems 3.2-3.6. We obtain conclusions of Theorems 3.2-3.6 by using Proposition 3.1, Theorem 3.1, Lemma 3.1, Lemma 3.2 and (1.14).□

Proofs of Theorems 3.7, 3.9 and 3.10. We see from Proposition 3.1, Theorem 3.1 and Dini's Theorem that $\boldsymbol{m}(\tilde{V}, \varepsilon^2)$ is continuous on $\overline{\mathcal{G}} \setminus \{(1,0)\}$ by defining

$$\mathcal{G}\left(\tilde{V}, \frac{\tilde{V}}{\pi^2}\right) := \tilde{V} + 1 \quad (\tilde{V} \ge 0),$$

$$\mathcal{G}(\tilde{V}, 0) := \begin{cases} 2\tilde{V} + 1 \quad (0 \le \tilde{V} < 1), \\ 2 \quad (\tilde{V} = 1), \\ \tilde{V} \quad (\tilde{V} > 1), \end{cases}$$

where

$$\overline{\mathcal{G}} := \left\{ (\tilde{V},\ \varepsilon^2) : \tilde{V} \geq 0,\ 0 \leq \varepsilon^2 \leq \frac{\tilde{V}}{\pi^2} \right\}.$$

Thus we obtain conclusions of Theorems 3.7, 3.9 and 3.10.□

We will give a proof of Theorem 3.8 in Section 8.

4 All exact solutions for $(AP;\tilde{V})$

In this section, we show all exact solutions for $(AP;\tilde{V})$ and an expression of $\boldsymbol{m}(\tilde{V},\varepsilon^2)$ defined in Section 1.

4.1 Representation formula

We show two theorems.

Theorem 4.1 *Let $\tilde{V}>0$. There exists a solution of (AP;$\tilde{V}$), if and only if $(\tilde{V},\ \varepsilon^2)\in\mathcal{G}$, where $\mathcal{G}$ is defined by (1.8). Moreover, the solution is unique and it has properties (1.9) and (1.10).*

The solution $W(x;\tilde{V},\varepsilon^2)$ is represented by

$$W(x,\tilde{V},\varepsilon^2)=\frac{\tilde{V}+2}{3}$$
$$+\frac{1}{\sqrt{3}}\sqrt{\tilde{V}^2+\tilde{V}+1}\cdot\frac{\beta\cdot(1-hs)\mathrm{sn}^2(K(\sqrt{h})x,\sqrt{h})+\alpha\cdot\mathrm{cn}^2(K(\sqrt{h})x,\sqrt{h})}{(1-hs)\mathrm{sn}^2(K(\sqrt{h})x,\sqrt{h})+\mathrm{cn}^2(K(\sqrt{h})x,\sqrt{h})},\tag{4.1}$$

$$\alpha:=\alpha(h,s)=\frac{3hs^2-2(1+h)s+1}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}},\tag{4.2}$$

$$\beta:=\beta(h,s)=\frac{-hs^2-2(1-h)s+1}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}},\tag{4.3}$$

where $(h,s)=\Big(h(\tilde{V},\varepsilon^2),s(\tilde{V},\varepsilon^2)\Big)$ is the unique solution of the following system of transcendental equations

$$(\mathrm{E})\begin{cases}\mathcal{E}(h,s)=\sqrt{3}\cdot\dfrac{\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}, & (4.4)\\[2mm] \mathcal{A}(h,s)=\dfrac{1}{3\sqrt{3}}\cdot\dfrac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\left(\sqrt{\tilde{V}^2+\tilde{V}+1}\right)^3}, & (4.5)\\[2mm] 0<h<1,\quad 0<s<1, & (4.6)\end{cases}$$

where

$$\mathcal{E}(h,s):=\frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(1+h)s+3}},\tag{4.7}$$

$$\mathcal{A}(h,s):=\frac{2(hs^2-2sh+1)(hs^2-2s+1)(1-hs^2)}{\left(\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}\right)^3}.\tag{4.8}$$

Here, $\mathrm{sn}(\cdot,\cdot)$, $\mathrm{cn}(\cdot,\cdot)$ *are Jacobi's elliptic functions.*

Theorem 4.2 *Let $W(x;\tilde{V},\varepsilon^2)$ be the unique solution of (AP;$\tilde{V}$), and $\boldsymbol{m}(\tilde{V},\ \varepsilon^2)$ is defined by (1.12). Then (1.13) and (1.14) holds.*

Moreover, it holds that

$$\boldsymbol{m}(\tilde{V},\ \varepsilon^2) = \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2+\tilde{V}+1} \cdot \mathcal{M}(h,s), \tag{4.9}$$

$$\mathcal{M}(h,s) := \frac{-(hs^2-2(1+h)s+3)+4(1-s)(1-sh)\Pi(-sh,\sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(1+h)s+3}}, \tag{4.10}$$

where $h = h(\tilde{V},\varepsilon^2)$, $s = s(\tilde{V},\varepsilon^2)$ are given in Theorem 4.1. Here, $\Pi(\cdot,\cdot)$ is the complete elliptic integral of the third kind.

We show graphs of $\mathcal{A}(h,s)$ and $\mathcal{E}(h,s)$ with level curves in Figures 4.1 and 4.2.

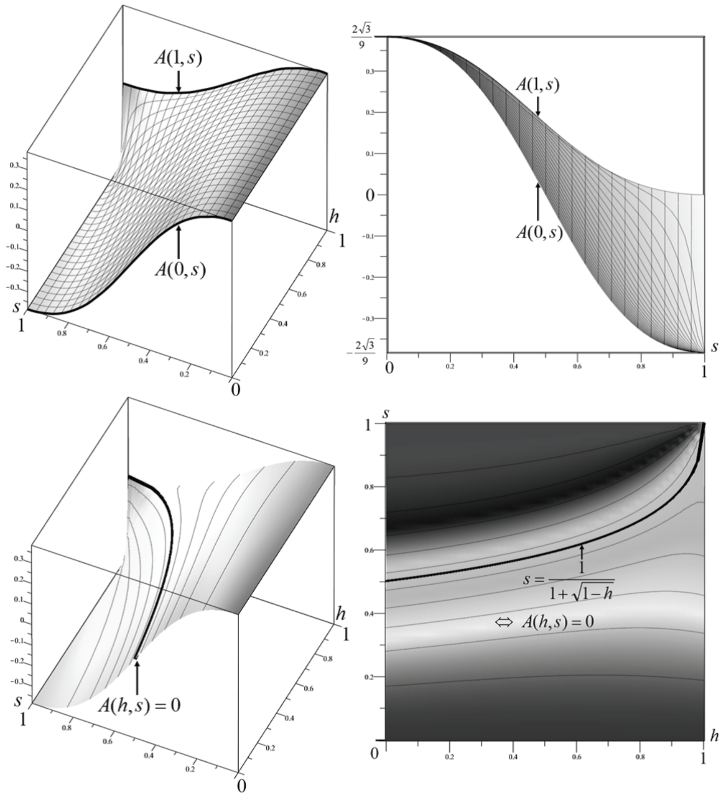

Figure 4.1: Graph and level curves of $\mathcal{A}(h,s)$

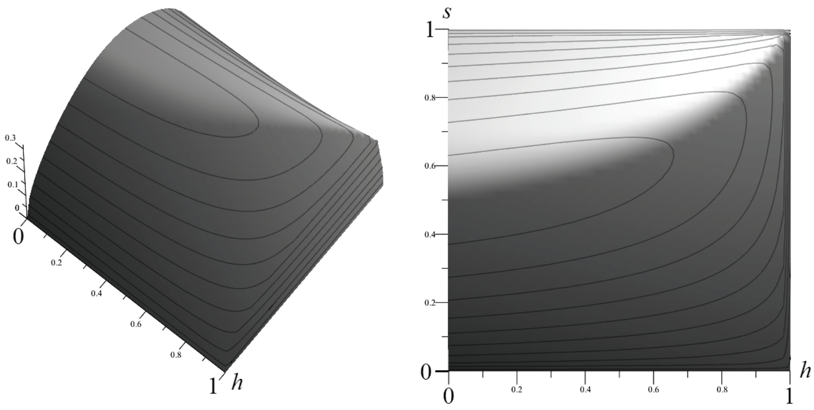

Figure 4.2: Graph and level curves of $\mathcal{E}(h,s)$

4.2 Proofs of Theorems 4.1-4.2

We prepare several propositions to prove Theorems 4.1 and 4.2.

We will use results in Kosugi, Morita and Yotsutani[3]. We see from Proposition 1.1 and its proof in [3] that the following lemma holds.

Lemma 4.1 *Let $E > 0$ and A be constants. Then all the solution of*

$$\begin{cases} E^2 u_{xx} - u^3 + u - A = 0 \quad \text{in } (0,1), \\ u_x(0) = u_x(1) = 0, \\ u_x(x) > 0 \quad \text{in } (0,1) \end{cases}$$

are represented by two parameters (h,s) with $0 < h < 1$ and $0 < s < 1$ as follows.

$$u(x;h,s) = \frac{\beta\cdot(1-hs)\mathrm{sn}^2(K(\sqrt{h})x,\sqrt{h}) + \alpha\cdot\mathrm{cn}^2(K(\sqrt{h})x,\sqrt{h})}{(1-hs)\mathrm{sn}^2(K(\sqrt{h})x,\sqrt{h}) + \mathrm{cn}^2(K(\sqrt{h})x,\sqrt{h})}, \tag{4.11}$$

$$\alpha := \alpha(h,s), \quad \beta := \beta(h,s), \tag{4.12}$$

where $\alpha(h,s)$ and $\beta(h,s)$ are defined by (4.2) and (4.3), and (h,s) is a solution of the following system of transcendental equations

$$\begin{cases} \mathcal{E}(h,s) = E, & (4.13) \\ \mathcal{A}(h,s) = A, & (4.14) \\ 0 < h < 1, \quad 0 < s < 1, & (4.15) \end{cases}$$

where $\mathcal{E}(h,s)$ and $\mathcal{A}(h,s)$ are defined by (4.8) and (4.7) respectively.

Moreover,

$$\int_0^1 u(x)dx = \mathcal{M}(h,s), \tag{4.16}$$

where $\mathcal{M}(h,s)$ is defined (4.10).

Proposition 4.1 *Let $W(x)$ be a solution of (AP;$\tilde{V}$), and*

$$u(x) := \frac{\sqrt{3}}{\sqrt{\lambda^2-\lambda+1}}\left(\lambda\, W(x) - \left(\frac{1}{3}+\frac{\lambda}{3}\right)\right), \tag{4.17}$$

where

$$\lambda := \frac{1}{\tilde{V}+1}. \tag{4.18}$$

Then $u(x)$ satisfies

$$\begin{cases} \left(\frac{\sqrt{3}\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}\right)^2 u_{xx} - u^3 + u \\ \qquad + \frac{1}{3\sqrt{3}} \cdot \frac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\left(\sqrt{\tilde{V}^2+\tilde{V}+1}\right)^3} = 0 \quad \text{in } (0,1), & (4.19) \\ u_x(0) = u_x(1) = 0, & (4.20) \\ u_x(x) > 0 \quad \text{in } (0,1), & (4.21) \end{cases}$$

and

$$\int_0^1 W(x)dx = \frac{\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2+\tilde{V}+1} \int_0^1 u(x)dx. \tag{4.22}$$

Proof. Let us put

$$U(x) := \frac{W(x)}{\tilde{V}+1}.$$

We get

$$\begin{cases} (\varepsilon\lambda)^2 U_{xx} + U(1-U)(U-\lambda) = 0 \quad \text{in } (0,1), \\ U_x(0) = U_x(1) = 0, \\ U_x(x) > 0 \quad \text{in } (0,1), \end{cases}$$

and

$$\int_0^1 W(x)dx = \frac{1}{\lambda}\int_0^1 U(x)dx,$$

where $\lambda = 1/(\tilde{V}+1)$.

We further introduce $u(x)$ by

$$u(x) := \frac{\left(U(x) - \left(\frac{1}{3}+\frac{\lambda}{3}\right)\right)}{c}, \quad c := \frac{\sqrt{\lambda^2-\lambda+1}}{\sqrt{3}}.$$

We have

$$U(x) = cu(x) + \frac{1}{3} + \frac{\lambda}{3},$$

and obtain

$$\begin{cases} \left(\frac{\lambda\varepsilon}{c}\right)^2 u_{xx} - u^3 + u + \frac{1}{3\sqrt{3}} \frac{(\lambda-2)(2\lambda-1)(\lambda+1)}{\left(\sqrt{\lambda^2-\lambda+1}\right)^3} = 0 \quad \text{in } (0,1), \\ u_x(0) = u_x(1) = 0, \\ u_x(x) > 0 \quad \text{in } (0,1), \end{cases}$$

and

$$\int_0^1 W(x)dx = \frac{1}{\lambda}\left(c\int_0^1 u dx + \frac{1+\lambda}{3}\right).$$

Hence, we get

$$\begin{cases} \left(\frac{\sqrt{3}\lambda\varepsilon}{\sqrt{\lambda^2-\lambda+1}}\right)^2 u_{xx} - u^3 + u \\ \qquad + \frac{1}{3\sqrt{3}} \frac{(\lambda-2)(2\lambda-1)(\lambda+1)}{\left(\sqrt{\lambda^2-\lambda+1}\right)^3} = 0 \quad \text{in } (0,1), \\ u_x(0) = u_x(1) = 0, \\ u_x(x) > 0 \quad \text{in } (0,1), \end{cases}$$

and

$$\int_0^1 W(x)dx = \frac{1}{\lambda}\left(\frac{\sqrt{\lambda^2-\lambda+1}}{\sqrt{3}}\int_0^1 u(x)dx + \frac{1+\lambda}{3}\right).$$

Therefore, we obtain

$$\begin{cases} \left(\frac{\sqrt{3}\varepsilon}{\sqrt{\tilde{V}^2+\tilde{V}+1}}\right)^2 u_{xx} - u^3 + u \\ \qquad - \frac{1}{3\sqrt{3}} \cdot \frac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\left(\sqrt{\tilde{V}^2+\tilde{V}+1}\right)^3} = 0 \quad \text{in } (0,1), \\ u_x(0) = u_x(1) = 0, \\ u_x(x) > 0 \quad \text{in } (0,1), \end{cases}$$

and

$$\int_0^1 W(x)dx = \frac{\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2+\tilde{V}+1} \int_0^1 u(x)dx.$$

□

The following proposition immediately follows from Lemma 4.1 and Proposition 4.1.

Proposition 4.2 *Let $\tilde{V} > 0$. There exists a solution $W(x)$ of (AP;$\tilde{V}$), if and only if (E) has a solution (h, s). For the solution (h, s) of (E), (AP;$\tilde{V}$) has a solution in the form (4.1) with (4.2) and (4.3).*

The following proposition is crucial for the proof of Theorem 4.1. We will give a proof of it in Subsection 4.3.

Proposition 4.3 *Let $\tilde{V} > 0$. There exists a solution $(h, s) = (h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2))$ of (E), if and only if $(\tilde{V}, \varepsilon^2) \in \mathcal{G}$, where $\mathcal{G}$ is defined by (1.8). Moreover, the solution is unique.*

We obtain following proposition by Lemma 4.1 and Proposition 4.1.

Proposition 4.4 *Let $\tilde{V} > 0$, $\varepsilon > 0$, $(h, s) = (h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2))$ be the unique solution of (E), $W(x; \tilde{V}, \varepsilon^2)$ be the unique solution of (E) in the form (4.1) with (4.2) and (4.3), and $u(x)$ be defined by (4.17) and (4.18) with $W(x) = W(x; \tilde{V}, \varepsilon^2)$. Then*

$$\int_0^1 u(x)dx = \mathcal{M}(h(\tilde{V}, \varepsilon^2), s(\tilde{V}, \varepsilon^2)), \tag{4.23}$$

where $\mathcal{M}(h, s)$ is defined by (4.10).

Now, we give a proofs of Theorems 4.1 and 4.2.

Proof of Theorem 4.1. We see from Propositions 4.2 and 4.3 that conclusions hold except (1.10).

We see that

$$\tilde{V} + 1 - \tilde{V} \cdot W\left(1 - x; \frac{1}{\tilde{V}}, \frac{\varepsilon^2}{\tilde{V}^2}\right)$$

is a solution of (AP; $\tilde{V}$). Thus, we obtain (1.10) by the uniqueness of solutions of $(AP; \tilde{V})$. □

Proof of Theorem 4.2. We obtain conclusions by (1.13), Propositions 4.1, and 4.4. □

4.3 Proof of Proposition 4.3

We have the following lemma by Lemma 3.2 and the proof of Lemma 3.4 in [3].

Lemma 4.2 *Let $\mathcal{E}(h, s)$ be defined by (4.7). The derivative of $\mathcal{E}(h, s)$ with respect to s satisfies*

$$\frac{\partial}{\partial s}\mathcal{E}(h, s) \begin{cases} > 0, & s \in (0, \sigma(h)), \quad h \in [0, 1), \\ = 0, & s = \sigma(h), \quad h \in [0, 1), \\ < 0, & s \in (\sigma(h), 1), \quad h \in [0, 1), \end{cases} \tag{4.24}$$

where $\sigma(h) := 1/(1+\sqrt{1-h})$. *Moreover,*

$$\mathcal{E}(h,\sigma(h)) = \frac{1}{\sqrt{2(2-h)}\,K(\sqrt{h})}, \tag{4.25}$$

$$\frac{d}{dh}\mathcal{E}(h,\sigma(h)) < 0 \quad \text{for } h \in [0,1), \tag{4.26}$$

and

$$\mathcal{E}(0,\sigma(0)) = \frac{1}{\pi}, \quad \mathcal{E}(h,\sigma(h)) \to 0 \text{ as } h \ \to 1. \tag{4.27}$$

In addition,

$$\mathcal{E}(0,s) = \frac{2\sqrt{2\,s\,(1-s)}}{\pi\sqrt{4\,s^2-4\,s+3}}. \tag{4.28}$$

We show graphs of $\partial\mathcal{E}(h,s)/\partial s$ in Figure 4.3.

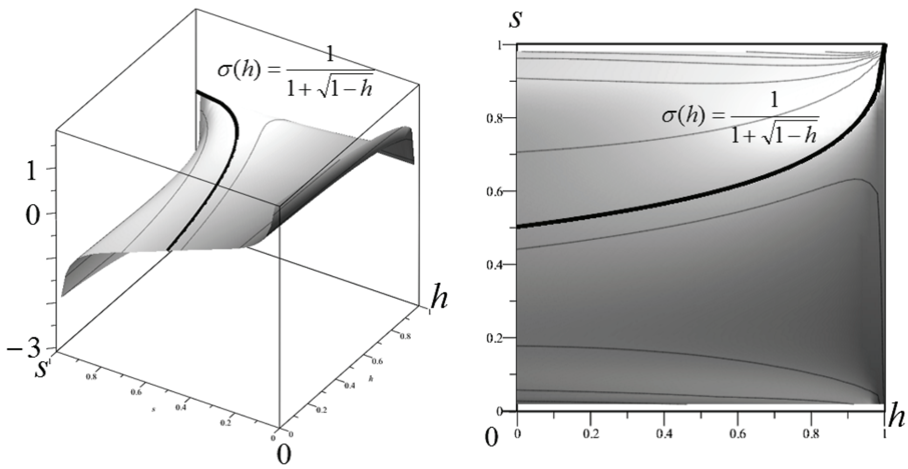

Figure 4.3: Graph of $\partial\mathcal{E}(h,s)/\partial s$

We have following lemmas.

Lemma 4.3 *Let*

$$r(v) := \frac{\sqrt{3}}{9}\cdot\frac{(1-v)(2v+1)(v+2)}{\left(\sqrt{v^2+v+1}\right)^3}. \tag{4.29}$$

Then $r(v)$ *is monotone decreasing in* $(0,\infty)$ *and*

$$r(0) = \frac{2\sqrt{3}}{9}, \quad r(1) = 0, \quad r(v) \to -\frac{2\sqrt{3}}{9} \text{ as } v \to \infty. \tag{4.30}$$

Proof. It is obvious from

$$\frac{dr(v)}{dv} = -\frac{3\sqrt{3}}{2} \cdot \frac{v(v+1)}{\left(\sqrt{v^2+v+1}\right)^5}. \tag{4.31}$$

□

We show graph of $r(v)$ in Figure 4.4.

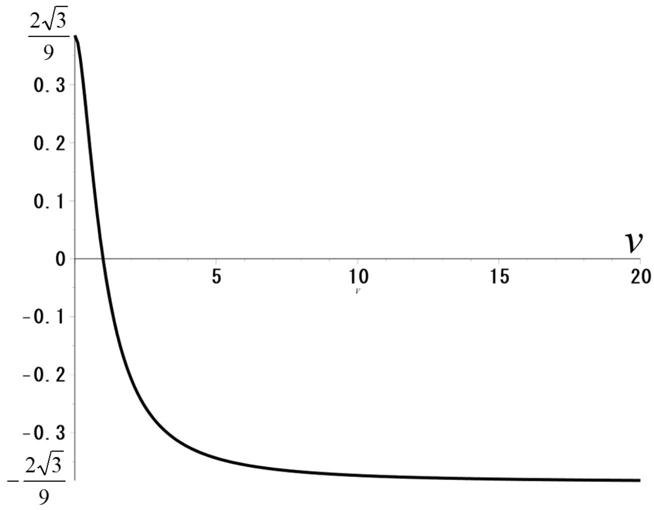

Figure 4.4: Graph of $r(v)$

Lemma 4.4 *Let $\mathcal{A}(h,s)$ be defined by (4.8). Then*

$$\mathcal{A}(h,0) = \frac{2\sqrt{3}}{9}, \quad \mathcal{A}(h,1) = -\frac{2\sqrt{3}}{9} \quad \text{for all } h \in [0,1), \tag{4.32}$$

$$\mathcal{A}_s(h,s) < 0 \quad \text{for all } (h,s) \in (0,1) \times (0,1). \tag{4.33}$$

We show graphs of $\mathcal{A}_s(h,s)$ in Figure 4.5.

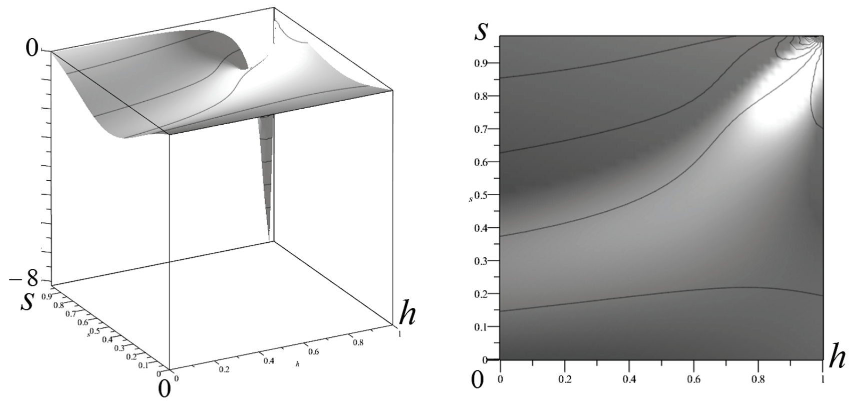

Figure 4.5: Graph of $\mathcal{A}_s(h,s)$

Proof. It is easy to see (4.32).

We will show (4.33). We have

$$\mathcal{A}_s = -\frac{32s\,(1-s)\,(1-hs)\cdot f(h,s)}{\left(s^2\,(3\,s^2-4\,s+4)\,h^2-2\,s\,(2\,s^2-s+2)\,h+4\,s^2-4\,s+3\right)^{5/2}}, \tag{4.34}$$

where

$$\begin{aligned} f(h,s) := {} & \left(h^2-h+1\right)h^2s^4-2\,(h+1)\,h^2s^3+6h^2s^2 \\ & -2\,(h+1)\,hs+h^2-h+1. \end{aligned} \tag{4.35}$$

We may show that

$$f(h,s) > 0 \quad \text{for all} \quad (h,s) \in (0,1)\times(0,1). \tag{4.36}$$

We show graphs of $f(h,s)$ in Figure 4.6.

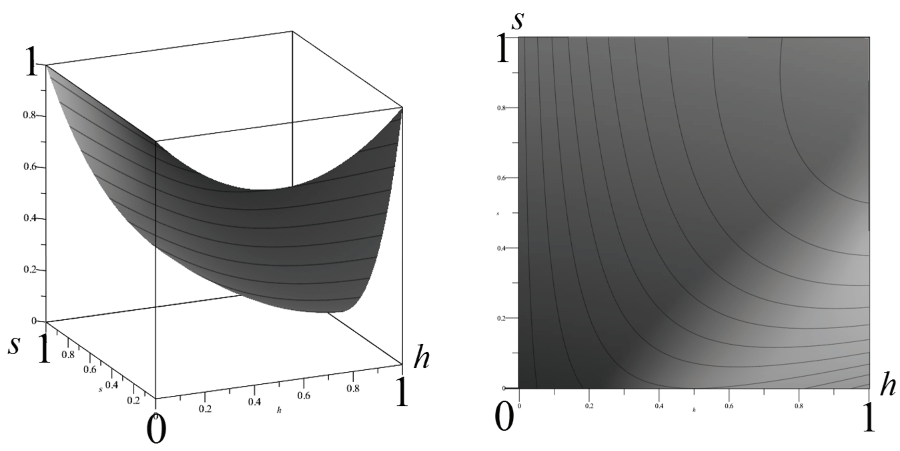

Figure 4.6: Graph of $f(h,s)$

We will show that $f(h,s)$ has no critical point in $(0,1)\times(0,1)$. We get

$$\begin{aligned}
f_h(h,s) &= -4\,h^3s^4+3\,h^2s^4+6\,h^2s^3-2\,hs^4+4\,hs^3-12\,hs^2\\
&\qquad +4\,hs-2\,h+2\,s+1,\\
f_s(h,s) &= -2\,h\left(2\,h^3s^3-2\,h^2s^3-3\,h^2s^2+2hs^3-3\,hs^2+6\,hs-h-1\right).
\end{aligned}$$

By virtue of the Euclid's algorithm, we have

$$\begin{aligned}
&\left(h^2-h+1\right)^2 f_0(h,s)\\
&=\left(\frac{1}{2}\left(h^2-h+1\right)\left(4h^2-3h+2\right)s+\frac{1}{4}\left(7h^2-7h-2\right)\right)\cdot f_1(h,s)\\
&\quad+\frac{1}{4}\left(1-h\right)\cdot f_2(h,s),
\end{aligned}$$

$$\begin{aligned}
&h\left(h^2+5h-2\right)^2 f_1(h,s)\\
&=\left(-\frac{2}{9}h\left(h^2-h+1\right)\left(h^2+5h-2\right)s\right.\\
&\quad\left.-\frac{16}{81}h^6+\frac{4}{9}h^5-\frac{43}{27}h^4+\frac{298}{81}h^3-\frac{1}{3}h^2-\frac{22}{27}h+\frac{8}{81}\right)\cdot f_2(h,s)\\
&\quad+\frac{8}{81}\cdot\left(h^2-h+1\right)^2\cdot f_3(h,s),
\end{aligned}$$

$$\begin{aligned}
&\left(16h^6-42h^5+117h^4-178h^3+75h^2+4\right)^2\cdot f_2(h,s)\\
&=\left(-9h\left(h^2+5h-2\right)\left(16h^6-42h^5+117h^4-178h^3+75h^2+4\right)s\right.\\
&\quad+128h^{10}-640h^9+1848h^8-4731h^7+4860h^6+1860h^5-1152h^4\\
&\quad\left.-2697h^3+1632h^2-100h-16\right)\cdot f_3(h,s)\\
&\quad-81h\cdot(h^2+5h-2)^2\cdot f_4(h),
\end{aligned}$$

where

$$\begin{aligned}
f_0(h,s) &:= f_h,\\
f_1(h,s) &:= 2h^3s^3-2h^2s^3-3h^2s^2+2hs^3-3hs^2+6hs-h-1,\\
f_2(h,s) &:= -9h\left(h^2+5h-2\right)s^2+\left(8h^4-10h^3+60h^2+2h-4\right)s\\
&\qquad -8h^4+12h^3-27h^2+h-6,\\
f_3(h,s) &:= \left(16h^6-42h^5+117h^4-178h^3+75h^2+4\right)s\\
&\qquad -16h^6+28h^5-133h^4+233h^3+h^2-79h+6,\\
f_4(h) &:= 32h^9-128h^8+376h^7-845h^6+730h^5\\
&\qquad +129h^4-227h^3-114h^2+77h+2. \qquad (4.37)
\end{aligned}$$

Thus, $f_h(h,s)=0$ and $f_s(h,s)=0$ implies

$$(h^2+5h-2)^2\cdot f_4(h)=0,\quad f_3(h,s)=0$$

note that

$$f_4(h)>0 \quad \text{for all} \quad h\in(0,1) \tag{4.38}$$

which we will show in Lemma 4.7 in this section.

Thus, we obtain

$$h^2+5h-2=0,\quad (-121824h+45360)s+114264h-42552=0,$$

which implies from $0<h<1$

$$h=\frac{-5+\sqrt{33}}{2},\quad s=\frac{17+\sqrt{33}}{12}>1.$$

This contradicts to $s\in(0,1)$. Therefore, $f(h,s)$ has no critical point in $(0,1)\times(0,1)$.

Let us check values of $f(h,s)$ on the boundary. We have

$$f(h,0)=h^2-h+1>0,\quad f(h,1)=(h^2-h+1)(1-h)^2>0,$$
$$f(0,s)=1>0,\quad f(1,s)=(1-s)^4>0,$$

for $0<h<1,\ 0<s<1$. Thus we complete the proof. □

Lemma 4.5 *Let $\tilde{V}>0$ be fixed. There exists a unique curve*

$$s(h;\tilde{V})\in C([0,1))\cap C^\infty((0,1)) \tag{4.39}$$

such that

$$\mathcal{A}(h,s(h;\tilde{V}))=\frac{1}{3\sqrt{3}}\cdot\frac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{\left(\sqrt{\tilde{V}^2+\tilde{V}+1}\right)^3},\quad 0<s(h;\tilde{V})<1. \tag{4.40}$$

Moreover,

$$s(0;\tilde{V})=\frac{1}{2}-\frac{1-\tilde{V}}{\sqrt{2}\sqrt{(\tilde{V}+2)(2\tilde{V}+1)}}, \tag{4.41}$$

and

$$s(h;1)=\frac{1}{1+\sqrt{1-h}}\quad(0<h<1). \tag{4.42}$$

Proof. We obtain the existence and uniqueness of $s(h;\tilde{V})$ by Lemmas 4.3 and 4.4. The assertion (4.39) is obtained by employing standard implicit function theorems.

We will show (4.41). By the construction of $s(0;\tilde{V})$, it is the unique solution of

$$\frac{2(-2s+1)}{(4s^2-4s+3)^{3/2}} = \frac{1}{3\sqrt{3}} \cdot \frac{(1-\tilde{V})(2\tilde{V}+1)(\tilde{V}+2)}{(\tilde{V}^2+\tilde{V}+1)^{3/2}}.$$

We can obtain the solution exactly, and get (4.41).

We will show (4.42). By the equation (4.40), We obtain $\mathcal{A}(h, s(h;1)) = 0$. We can obtain the solution exactly, and get (4.42). □

Let us show that $\mathcal{E}(h, s(h;\tilde{V}))$ is decreasing in h.

Lemma 4.6 *Let $\mathcal{E}(h,s)$ be defined by (4.7), and $s(h;\tilde{V})$ defined in Lemma 4.5, then for each fixed $\tilde{V} > 0$,*

$$\mathcal{E}(0, s(0;\tilde{V})) = \frac{\sqrt{3}\sqrt{\tilde{V}}}{\pi\sqrt{\tilde{V}^2+\tilde{V}+1}}, \tag{4.43}$$

$$\mathcal{E}(h, s(h;\tilde{V})) \to 0 \text{ as } h \to 1, \tag{4.44}$$

and

$$\frac{d\mathcal{E}(h, s(h;\tilde{V}))}{dh} < 0 \quad \text{in } (0,1). \tag{4.45}$$

We show graphs of $d\mathcal{E}(h, s(h;\tilde{V}))/dh$ in Figure 4.7.

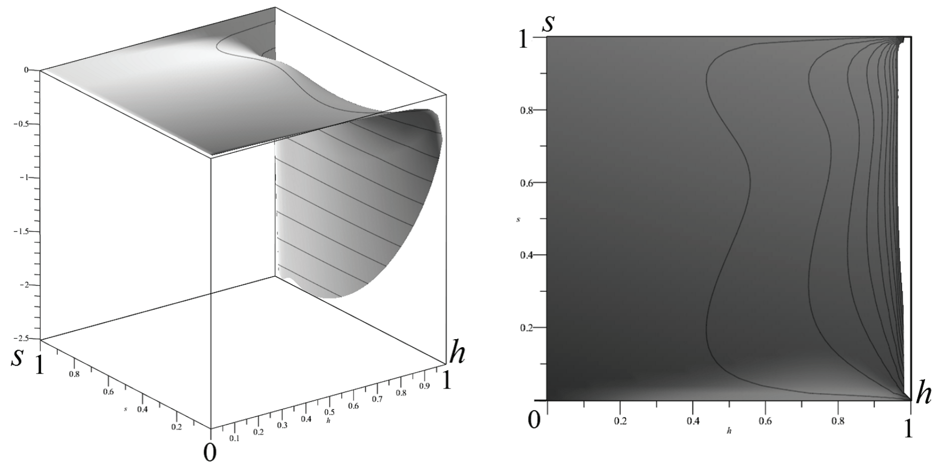

Figure 4.7: Graph of $d\mathcal{E}(h, s(h;\tilde{V}))/dh$

Proof. We obtain (4.43) by (4.41) and

$$\mathcal{E}(0,s) = 2\frac{\sqrt{2}\sqrt{s(1-s)}}{\pi\sqrt{4s^2-4s+3}}. \tag{4.46}$$

We obtain (4.44) by Lemma 4.2.

We will show (4.45). Let us denote $s(h;\tilde{V})$ by $s(h)$ or s, since $\tilde{V}$ is given and fixed.

It holds that

$$\frac{d\mathcal{E}(h,s(h))}{dh} = \mathcal{E}_h + \mathcal{E}_s \cdot \frac{ds(h)}{dh}$$

and

$$\mathcal{A}_h + \mathcal{A}_s \cdot \frac{ds(h)}{dh} = 0.$$

Hence, we get

$$\frac{d\mathcal{E}(h,s(h))}{dh} = \frac{\mathcal{A}_s\mathcal{E}_h - \mathcal{E}_s\mathcal{A}_h}{\mathcal{A}_s}. \tag{4.47}$$

We have (4.34),

$$\begin{aligned}\mathcal{A}_h = &-16s^2(1-s)^2(1-hs)\left\{s^3h^2+\left(-2s^3+3s^2-3s+2\right)h-1\right\}\\ &/\left(\left(3s^4-4s^3+4s^2\right)h^2+\left(-4s^3+2s^2-4s\right)h+4s^2-4s+3\right)^{5/2},\end{aligned}$$

$$\begin{aligned}\mathcal{E}_s = &\,3\left(hs^2-2hs+1\right)\left(hs^2-2s+1\right)\left(1-hs^2\right)\\ &/\Big(\sqrt{2s(1-s)(1-hs)}K(\sqrt{h})\\ &\cdot\left(\left(3s^4-4s^3+4s^2\right)h^2+\left(-4s^3+2s^2-4s\right)h+4s^2-4s+3\right)^{3/2}\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{E}_h = \Big(&s(1-s)(1-hs)\\ &\cdot\left(\left(3s^4-4s^3+4s^2\right)h^2+\left(-4s^3+2s^2-4s\right)h+4s^2-4s+3\right)E(\sqrt{h})\\ &-s(1-s)(1-h)\\ &\cdot\left(\left(s^4+2s^3\right)h^2+\left(-8s^3+8s^2-6s\right)h+4s^2-4s+3\right)K(\sqrt{h})\Big)\\ &/\Big(-h(1-h)\sqrt{2s(1-s)(1-hs)}K(\sqrt{h})^2\\ &\quad\cdot\left(\left(3s^4-4s^3+4s^2\right)h^2+\left(-4s^3+2s^2-4s\right)h+4s^2-4s+3\right)^{3/2}\Big).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\mathcal{A}_s\mathcal{E}_h - \mathcal{E}_s\mathcal{A}_h =\\ &\left(8\sqrt{2}(1-hs)^2s^2(1-s)^2\right)\left(2f(h,s)E(\sqrt{h})-P(h,s)K(\sqrt{h})\right)\\ &/\Big((1-h)h\left(\left(3s^4-4s^3+4s^2\right)h^2+\left(-4s^3+2s^2-4s\right)h+4s^2-4s+3\right)^3\\ &\quad\cdot K(\sqrt{h})^2\sqrt{s(1-s)(1-hs)}\Big),\end{aligned} \tag{4.48}$$

where $f(h,s)$ is defined by (4.35),

$$P(h,s) := s^4h^4 + \left(-3s^4 + 4s^3 - 6s^2\right)h^3 + \left(2s^4 - 4s^3 + 6s^2 + 4s + 1\right)h^2 + (-4s-3)h + 2. \tag{4.49}$$

We show graphs of $P(h,s)$ in Figure 4.8.

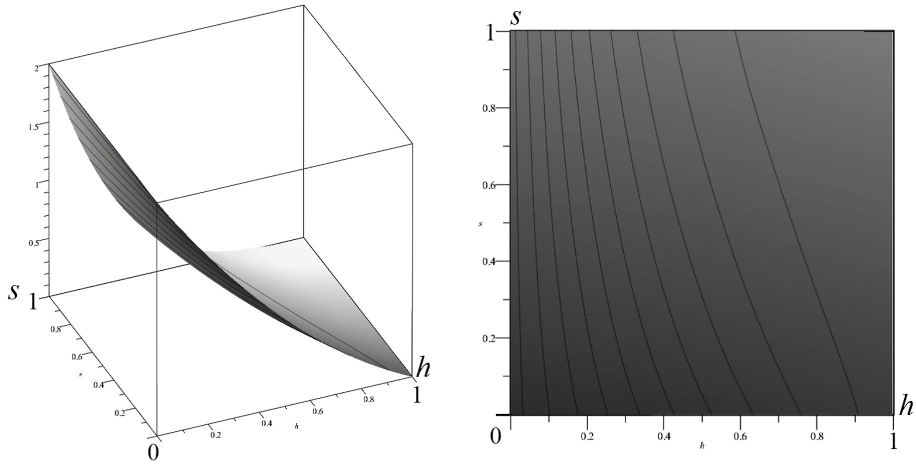

Figure 4.8: Graph of $P(h,s)$

We note that (4.36) holds, and

$$E(\sqrt{h}) - K(\sqrt{h})\sqrt{1-h} > 0 \quad \text{for } h \in (0,1),$$

which is easy to prove by differentiation.

Thus we may show

$$2f(h,s)\sqrt{1-h} - P(h,s) > 0 \quad \text{in } (0,1)\times(0,1). \tag{4.50}$$

Now let us put $H := \sqrt{1-h}$, that is, $h := 1 - H^2$. We get

$$\begin{aligned} f(1-H^2,s) &= -s^4H^8 + \left(3s^4 - 2s^3\right)H^6 + \left(-4s^4 + 8s^3 - 6s^2 + 2s - 1\right)H^4 \\ &\quad + \left(3s^4 - 10s^3 + 12s^2 - 6s + 1\right)H^2 - s^4 + 4s^3 - 6s^2 + 4s - 1, \\ P(1-H^2,s) &= s^4H^8 + \left(-s^4 - 4s^3 + 6s^2\right)H^6 \\ &\quad + \left(-s^4 + 8s^3 - 12s^2 + 4s + 1\right)H^4 \\ &\quad + \left(s^4 - 4s^3 + 6s^2 - 4s + 1\right)H^2. \end{aligned}$$

Therefore, we obtain

$$2f(1-H^2,s)H - P(1-H^2,s) = H\,(1-H)^2\,F(H,s),$$

where

$$F(H,s) := \left(2H^2+3H+2\right)(1-H)^2(H+1)^2 s^4 - 4(H+2)(1-H)(H+1)^2 s^3 + 6(2-H)(H+1)^2 s^2 - 4(H+2)(H+1)s + 2H^2+3H+2 \tag{4.51}$$

We show that

$$F(H,s) > 0 \quad \text{in } (0,1)\times(0,1). \tag{4.52}$$

in Lemma 4.8. Thus we complete the proof.□

We show proofs of Lemma 4.7 and 4.8 which we used in the proofs of Lemma 4.4 and 4.5, respectively.

Lemma 4.7 *Let $f_4(h)$ be defined by (4.37) then (4.38) holds.*

We show graphs of $f_4(h)$ in Figures 4.9.

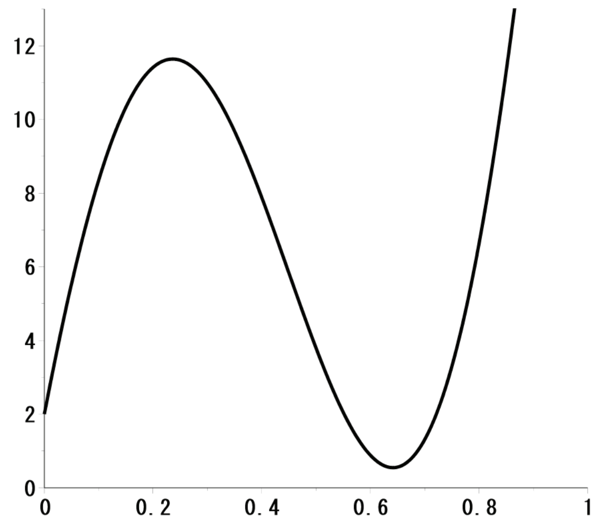

Figure 4.9: Graph of $f_4(h)$

Proof. We obtain a Sturm's sequence for f_4 as

$$
\begin{aligned}
&sts(h) := \\
&\Big[h^9 - 4h^8 + \frac{47}{4}h^7 - \frac{845}{32}h^6 + \frac{365}{16}h^5 + \frac{129}{32}h^4 - \frac{227}{32}h^3 - \frac{57}{16}h^2 + \frac{77}{32}h + \frac{1}{16}, \\
&h^8 - \frac{32}{9}h^7 + \frac{329}{36}h^6 - \frac{845}{48}h^5 + \frac{1825}{144}h^4 + \frac{43}{24}h^3 - \frac{227}{96}h^2 - \frac{19}{24}h + \frac{77}{288}, \\
&-h^7 + \frac{12287}{2672}h^6 - \frac{375}{167}h^5 - \frac{20405}{2672}h^4 + \frac{5097}{1336}h^3 + \frac{4953}{1336}h^2 - \frac{579}{334}h - \frac{235}{1336}, \\
&-h^6 + \frac{3125840}{1324671}h^5 - \frac{6762557}{9272697}h^4 - \frac{7518526}{9272697}h^3 \\
&+ \frac{183914}{9272697}h^2 + \frac{2201704}{9272697}h - \frac{66574}{9272697}, \\
&-h^5 + \frac{622141845373}{277036662631}h^4 - \frac{145639869262}{277036662631}h^3 \\
&- \frac{516971097161}{554073325262}h^2 + \frac{87880085542}{277036662631}h + \frac{28239941389}{554073325262}, \\
&-h^4 - \frac{50516367775371319}{42533564212819048}h^3 + \frac{19264738753473027}{2501974365459944}h^2 \\
&- \frac{180588941858621041}{42533564212819048}h + \frac{1110296803852637}{42533564212819048}, \\
&h^3 - \frac{92508599612088752293 83}{38258533947796915095 47}h^2 \\
&+ \frac{4442508415273413102621}{3825853394779691509547}h - \frac{437190502698420564 49}{3825853394779691509547}, \\
&-h^2 + \frac{538313218264902095585 3122}{109005509742970990644 19837}h + \frac{115467117719722177847 5355}{109005509742970990644 19837}, \\
&h + \frac{348181712337732739861 53872}{512593586578133768771 99485}, \quad 1\Big]
\end{aligned}
$$

Let us see Strum's sequence $sts(h)$ at $h = 0$ and $h = 1$. We have

$$
\begin{aligned}
sts(0) = \Big[&\frac{1}{16}, \frac{77}{288}, -\frac{235}{1336}, -\frac{66574}{9272697}, \frac{28239941389}{554073325262}, \\
&\frac{1110296803852637}{42533564212819048}, -\frac{437190502698420564 49}{3825853394779691509547}, \\
&\frac{115467117719722177847 5355}{109005509742970990644 19837}, \frac{348181712337732739861 53872}{512593586578133768771 99485}, 1\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
sts(1) = \Big[&1, \frac{5}{9}, -\frac{112}{167}, \frac{646144}{9272697}, \frac{42979821136}{277036662631}, \frac{6871497720760336}{5316695526602381}, \\
&-\frac{1026217201425612673664}{3825853394779691509547}, -\frac{436274761445085633009 1360}{109005509742970990644 19837}, \\
&-\frac{164411874240401028910 45613}{512593586578133768771 99485}, 1\Big],
\end{aligned}
$$

which implies $tsc(0) - tsc(1) = 4 - 4 = 0$. Thus, $f_4(h) = 0$ has no real root in $h \in (0, 1)$ by Strum's theorem, and (4.38) holds in view of $f_4(0) = 2 > 0$.□

Lemma 4.8 *Let $F(H, s)$ be defined by (4.51) then (4.52) holds.*

We show graphs of $F(H, s)$ in Figures 4.10.

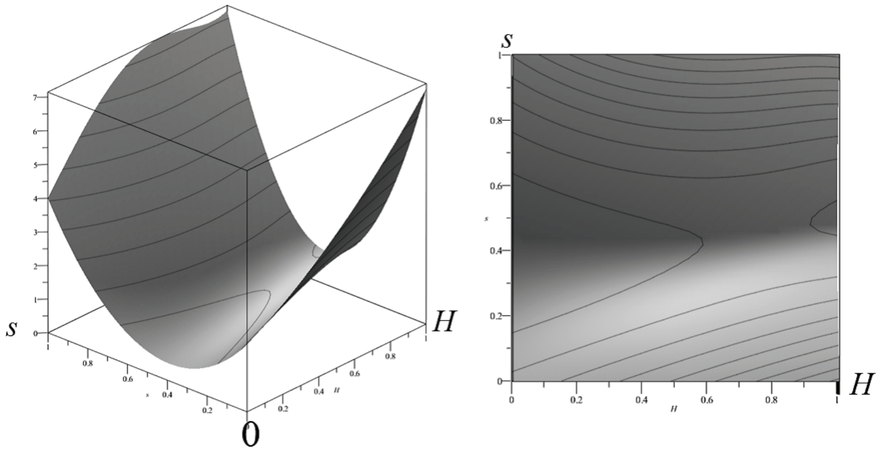

Figure 4.10: Graph of $F(H, s)$

Proof. It hold that

$$
\begin{aligned}
F_0 =& \frac{1}{4} \cdot \frac{(1-H)\left(2H^2+3H+2\right)s + H + 2}{(1-H)\left(2H^2+3H+2\right)} \cdot F_1 - F_2, \\
F_1 =& \frac{2}{3} \cdot \left(\frac{(1-H)^2 \left(2H^2+3H+2\right)^2 s}{H^3} \right. \\
&+ \left. \frac{\left(2H^2+3H+2\right)(1-H)(H+2)\left(2H^2-1\right)}{(H+1)\,H^3} \right) \cdot F_2 - F_3, \\
F_2 =& \frac{9}{2} \cdot \frac{H^2 (H+1)\left(\left(H^2-1\right)s + 2\,H + 1\right)}{(1-H)\left(2\,H^2+3\,H+2\right)^2 (2\,H+1)(H+2)} \cdot F_3 - F_4,
\end{aligned}
$$

where

$$F_0 := F,$$
$$F_1 := F_s = 4\,(H{+}1)\Big(\left(2H^2{+}3H{+}2\right)(H{+}1)\,(1{-}H)^2\,s^3$$
$$-3\,(H{+}2)\,(H{+}1)\,(1{-}H)\,s^2{+}3\,(H{+}1)\,(2{-}H)\,s{-}(H{+}2)\Big),$$
$$F_2 := \frac{2\left(3\,(1-H)\,(H+1)^2\,s^2-3\,(H+1)\,(H+2)\,s+2H^2+4H+3\right)H^3}{(1-H)\left(2H^2+3H+2\right)},$$
$$F_3 := -\frac{4}{3}\cdot\frac{(2H+1)\,(H+2)\left(2H^2+3H+2\right)H\,((H+1)s-1)}{H+1},$$
$$F_4 := \frac{4H^4}{2H^2+3H+2}.$$

Thus, we obtain a Strum's sequence for F in s as

$$STS(s;H) := [F_0, F_1, F_2, F_3, F_4].$$

We note that,

$$F_0(H,0) = 2H^2+3H+2 > 0,$$

Let us see Strum's sequence $STS(s;H)$ at $s=0$ and $s=1$. We have

$$STS(0;\ H) :=$$
$$\Bigg[2H^2+3H+2,\ -4(H+1)(H+2), \frac{2H^3\left(2H^2+4H+3\right)}{(1-H)\left(2H^2+3H+2\right)},$$
$$\frac{4}{3}\cdot\frac{H\,(2H+1)\,(H+2)\left(2H^2+3H+2\right)}{H+1},\ \frac{4H^4}{2H^2+3H+2}\Bigg]$$

and

$$STS(1;\ H) :=$$
$$\Bigg[H^4\left(2H^2{+}3H{+}2\right),\ 4H^4(H+1)(2H+1), -\frac{2H^4\left(3H^2+4H+2\right)}{(1-H)\left(2H^2+3H+2\right)},$$
$$-\frac{4}{3}\cdot\frac{H^2\,(2H+1)\,(H+2)\left(2H^2+3H+2\right)}{H+1},\ \frac{4H^4}{2H^2+3H+2}\Bigg]$$

Now, let us count times of sign changing $TSC(s)$. For $0 < H < 1$, we have
and $TSC(0) - TSC(1) = 2 - 2 = 0$, which implies that $F(H,s) = 0$ in $s \in (0,1)$ has no real root by Strum's theorem, and (4.52) holds.

Therefore, $F(H,s)$ has no critical point in $(0,1)\times(0,1)$. We may check values of the boundary. We have

s	F_0	F_1	F_2	F_3	F_4	$TSC(s)$
0	+	-	+	+	+	2
1	+	+	-	-	+	2

$$F(H,0) = 2H^2 + 3H + 2 > 0, \quad F(H,1) = H^4(2H^2 + 3H + 2) > 0,$$
$$F(0,s) = 2(s-1)^4 > 0, \quad F(1,s) = 24s^2 - 24s + 7 > 0,$$

for $0 < H < 1,\ 0 < s < 1$. Thus we complete the proof. □
Proof of Proposition 4.3. First, we note that

$$0 < \frac{\sqrt{3}\varepsilon}{\sqrt{\tilde{V}^2 + \tilde{V} + 1}} < \mathcal{E}(0, s(0; \tilde{V}))$$

is equivalent to

$$0 < \frac{\sqrt{3}\varepsilon}{\sqrt{\tilde{V}^2 + \tilde{V} + 1}} < \frac{\sqrt{3}\sqrt{\tilde{V}}}{\pi\sqrt{\tilde{V}^2 + \tilde{V} + 1}},$$

that is,

$$0 < \varepsilon < \frac{\sqrt{\tilde{V}}}{\pi}.$$

Thus we complete the proof by Lemma 4.6.□

5 Limit of $\boldsymbol{m}(\tilde{V},\varepsilon^2)$

In this section, we give a proof of Proposition 3.1, which show limits of $\boldsymbol{m}(\tilde{V},\varepsilon^2)$ as $\varepsilon^2 \to 0$ and $\varepsilon^2 \to \tilde{V}/\pi^2$.

5.1 Limit of $\boldsymbol{m}(\tilde{V},\varepsilon^2)$ as $\varepsilon^2 \to \tilde{V}/\pi^2$

We will show (i). Let $\tilde{V} \in (0,1)$ be fixed. We note that $(\tilde{V},\varepsilon^2) \to (\tilde{V},\tilde{V}/\pi^2)$ corresponds to

$$(h, s(h;\tilde{V})) \to (0, s(0;\tilde{V})) = \left(0, \frac{1}{2} - \frac{1-\tilde{V}}{\sqrt{2}\sqrt{(\tilde{V}+2)(2\tilde{V}+1)}}\right) \tag{5.1}$$

by Lemma 4.6 and (4.41). Hence we get $\boldsymbol{m}(\tilde{V},\varepsilon^2) \to \boldsymbol{m}(\tilde{V},\tilde{V}/\pi^2)$ as $\varepsilon^2 \to \tilde{V}/\pi^2$ and

$$\begin{aligned}
\boldsymbol{m}\left(\tilde{V}, \frac{\tilde{V}}{\pi^2}\right) &= \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2+\tilde{V}+1} \cdot \mathcal{M}(0, s(0;\tilde{V})) \\
&= \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2+\tilde{V}+1} \cdot \frac{1-2s(0;\tilde{V})}{\sqrt{4s(0;\tilde{V})^2 - 4s(0;\tilde{V})+3}} \\
&= \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2+\tilde{V}+1} \cdot \frac{1}{\sqrt{3}} \cdot \frac{1-\tilde{V}}{\sqrt{\tilde{V}^2+\tilde{V}+1}} \\
&= \tilde{V}+1
\end{aligned} \tag{5.2}$$

by (4.10) and (4.41). Thus we obtain (3.1) in view of (1.13) and (1.14).

5.2 Limit of $\boldsymbol{m}(\tilde{V},\varepsilon^2)$ as $\varepsilon^2 \to 0$

We will show (ii). Let $\tilde{V} \in (0,1)$ be fixed. We see that $(\tilde{V},\varepsilon) \to (\tilde{V},0)$ corresponds to

$$(h, s(h;\tilde{V})) \to (1, s(1;\tilde{V})) \tag{5.3}$$

by Lemma 4.6 and (4.40), where $s(1;\tilde{V})$ is the unique solution of

$$\mathcal{A}(1,s) = r(\tilde{V}), \tag{5.4}$$

and $0 < s(1;\tilde{V}) < 1$, where

$$\mathcal{A}(1,s) = \frac{2\,(1-s)^2\,(s+1)}{(3\,s^2 - 2\,s + 3)^{3/2}} \tag{5.5}$$

and $r(v)$ is defined by (4.29). In fact, it holds that

$$\mathcal{A}(1,0) = \frac{2\sqrt{3}}{9}, \quad \mathcal{A}(1,1) = 0, \quad 0 < r(\tilde{V}) < \frac{2\sqrt{3}}{9} \tag{5.6}$$

and $\mathcal{A}(1,s)$ is monotone decreasing in $s \in (0,1)$.

By solving (5.4), we obtain

$$9\tilde{V}(\tilde{V}+1)s^2 - 2(7\tilde{V}^2 + 7\tilde{V} + 4)s + 9\tilde{V}(\tilde{V}+1) = 0, \tag{5.7}$$

and

$$s(1;\tilde{V}) = \frac{7\,\tilde{V}^2 + 7\,\tilde{V} + 4 - 2(2\,\tilde{V}+1)\sqrt{(2\,\tilde{V}+4)(1-\tilde{V})}}{9\tilde{V}(\tilde{V}+1)}, \tag{5.8}$$

since

$$\begin{aligned}
&\mathcal{A}(1,s)^2 - r(\tilde{V})^2 \\
&= \frac{9\tilde{V}s^2 + (8\tilde{V}^2 + 2\tilde{V} + 8)s + 9\tilde{V}}{27(3s^2 - 2s + 3)^3(\tilde{V}^2 + \tilde{V} + 1)^3} \\
&\cdot ((9\tilde{V}+9)s^2 - (8\tilde{V}^2 + 14\tilde{V} + 14)s + 9\tilde{V} + 9) \\
&\cdot (9(\tilde{V}+1)\tilde{V}s^2 - 2(7\tilde{V}^2 + 7\tilde{V} + 4)s + 9\tilde{V}(\tilde{V}+1)) \\
&= \frac{9\tilde{V}s^2 + (8\tilde{V}^2 + 2\tilde{V} + 8)s + 9\tilde{V}}{27(3s^2 - 2s + 3)^3(\tilde{V}^2 + \tilde{V} + 1)^3} \\
&\cdot \left((9\,\tilde{V}+9)\left(s - \frac{1}{9}\,\frac{4\,\tilde{V}^2 + 7\,\tilde{V} + 7}{\tilde{V}+1} \right)^2 + \frac{(16\,\tilde{V}+8)(1-\tilde{V})(\tilde{V}+2)^2}{9\,\tilde{V}+9} \right) \\
&\cdot (9(\tilde{V}+1)\tilde{V}s^2 - 2(7\tilde{V}^2 + 7\tilde{V} + 4)s + 9\tilde{V}(\tilde{V}+1)).
\end{aligned} \tag{5.9}$$

We have

$$\begin{aligned}
&\lim_{h\to 1} \mathcal{M}(h, s(h,\tilde{V})) = \frac{s(1;\tilde{V}) + 1}{\sqrt{3s(1;\tilde{V})^2 - 2s(1;\tilde{V}) + 3}} \\
&= A(1, s(1;\tilde{V})) \cdot \frac{3s(1;\tilde{V})^2 - 2s(1;\tilde{V}) + 3}{2(1 - s(1;\tilde{V}))^2} \\
&= \frac{1}{6\sqrt{3}} \cdot \frac{(1-\tilde{V})(2\,\tilde{V}+1)(\tilde{V}+2)(3s(1;\tilde{V})^2 - 2s(1;\tilde{V}) + 3)}{(\tilde{V}^2 + \tilde{V} + 1)^{3/2}(1 - s(1;\tilde{V}))^2}
\end{aligned} \tag{5.10}$$

by $0 < s(1;\tilde{V}) < 1$ and

$$\lim_{h\to 1} \frac{(1 - hs(h;\tilde{V}))\Pi(-hs(h;\tilde{V}), \sqrt{h})}{K(\sqrt{h})} = 1. \tag{5.11}$$

Hence

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \boldsymbol{m}(\tilde{V},\varepsilon^2) = \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2+\tilde{V}+1} \cdot \lim_{h\to 1} \mathcal{M}(h, s(h,\tilde{V}))\\
&= \frac{4\tilde{V}+2}{3} + \frac{1}{\sqrt{3}} \cdot \sqrt{\tilde{V}^2+\tilde{V}+1}\\
&\cdot \left(\frac{1}{6\sqrt{3}} \cdot \frac{(1-\tilde{V})(2\,\tilde{V}+1)(\tilde{V}+2)(3s^2-2s+3)}{(\tilde{V}^2+\tilde{V}+1)^{3/2}(1-s)^2} \right) - (2\tilde{V}+1) + (2\tilde{V}+1)\\
&= -\frac{1}{18} \frac{(2\,\tilde{V}+1)\left(9\tilde{V}(\tilde{V}+1)s^2 - 2(7\tilde{V}^2+7\tilde{V}+4)s + 9\tilde{V}(\tilde{V}+1)\right)}{(\tilde{V}^2+\tilde{V}+1)\,(1-s)^2} + (2\tilde{V}+1)\\
&= 2\tilde{V}+1
\end{aligned} \tag{5.12}
$$

by (5.10) and (5.7), where $s = s(1;\tilde{V})$. Thus, we get (3.2).

We obtain (iii) by (ii) and (1.13).□

6 Monotonicity of $\boldsymbol{m}(\tilde{V}, \varepsilon^2)$

In this section, we give a proof of Theorem 3.1, which we show monotonicity of $\boldsymbol{m}(\tilde{V}, \varepsilon^2)$ in ε. We prepare several propositions to prove Theorem 3.1. We will give proofs of them in subsequent sections.

6.1 Proof of Theorem 3.1

The following properties shows the expression of $\partial \boldsymbol{m}(\tilde{V}, \varepsilon^2)/\partial \varepsilon$. Let $\mathcal{M}(h, s)$, $\mathcal{E}(h, s)$ and $\mathcal{A}(h, s)$ be defined by (4.10), (4.7) and (4.8) respectively, $s(h; \tilde{V})$ is defined in Lemma 4.5.

Proposition 6.1 *Let $\tilde{V} > 0$ be fixed. Then, $\boldsymbol{m}(\tilde{V}, \varepsilon^2)$ satisfies the following equation*

$$\frac{\partial \boldsymbol{m}(\tilde{V}, \varepsilon^2)}{\partial \varepsilon} = -\frac{\mathcal{M}_s \cdot \mathcal{A}_h - \mathcal{M}_h \cdot \mathcal{A}_s}{\mathcal{A}_s \cdot \dfrac{d\mathcal{E}(h, s(h; \tilde{V}))}{dh}}, \tag{6.1}$$

where

$$\begin{aligned}
&\mathcal{A}_s(h, s) \\
&= -32 \left(\mathcal{D}^{K2}\right)^{-5/2} s\,(1-s)\,(1-hs) \\
&\cdot \left(\left(h^2 - h + 1\right) h^2 s^4 - 2\,(h+1)\, h^2 s^3 + 6 h^2 s^2 - 2\,(h+1)\, hs + h^2 - h + 1\right),
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
&\mathcal{A}_h(h, s) \\
&= -16 \left(\mathcal{D}^{K2}\right)^{-5/2} s^2\,(1-s)^2\,(1-hs) \left(s^3 h^2 + \left(-2s^3 + 3s^2 - 3s + 2\right) h - 1\right),
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
&\mathcal{M}_s(h, s) \\
&= -2s^{-1} \left(\mathcal{D}^{K2}\right)^{-3/2} K(\sqrt{h})^{-1} \\
&\cdot \Big(-3\,\left(1 - hs^2\right)\left(hs^2 - 2hs + 1\right)\left(hs^2 - 2s + 1\right) \Pi(-hs, \sqrt{h}) + s \cdot \mathcal{D}^{K2} \cdot E(\sqrt{h}) \\
&+ \left(hs^2 - 2\,s + 1\right)\left(2\,h^2 s^3 - hs^3 - hs^2 - 4\,hs + 2\,s^2 - s + 3\right) K(\sqrt{h})\Big),
\end{aligned} \tag{6.4}$$

$$\begin{aligned}
&\mathcal{M}_h(h, s) \\
&= 2h^{-1}(1-h)^{-1} \left(\mathcal{D}^{K2}\right)^{-3/2} K(\sqrt{h})^{-2}\,(1-s) \Bigg(\Big(-\,(1-hs) \cdot \mathcal{D}^{K2} \cdot E(\sqrt{h}) \\
&+ (1-h)\left(h^2 s^4 + 2h^2 s^3 - 8hs^3 + 8hs^2 - 6hs + 4s^2 - 4s + 3\right) K(\sqrt{h})\Big) \\
&\cdot \Pi(-hs, \sqrt{h}) + \mathcal{D}^{K2} \cdot E(\sqrt{h}) K(\sqrt{h}) \\
&- (1-h)\left(h^2 s^4 - 4\,hs^3 + 4\,hs^2 - 4\,hs + 4\,s^2 - 4\,s + 3\right) K(\sqrt{h})^2\Bigg),
\end{aligned} \tag{6.5}$$

$$\mathcal{D}^{K2}(h,s) := \left(3s^4 - 4s^3 + 4s^2\right)h^2 + \left(-4s^3 + 2s^2 - 4s\right)h + 4\,s^2 - 4\,s + 3. \tag{6.6}$$

Moreover, it holds that

$$\mathcal{D}^{K2}(h,s) > 0 \quad (0 < h < 1, \;\; 0 < s < 1)\,, \tag{6.7}$$

$$\mathcal{A}_s \cdot \frac{d\mathcal{E}(h, s(h;\tilde{V}))}{dh} > 0. \tag{6.8}$$

Proof. Let $\tilde{V} > 0$ be fixed. Then $s = s(h;\tilde{V})$ by lemma 4.5. By the equation of (1.12), we get

$$\frac{\partial \boldsymbol{m}(\tilde{V},\varepsilon^2)}{\partial \varepsilon} = -\frac{\dfrac{d\mathcal{M}(h,s(h;\tilde{V}))}{dh}}{\sqrt{\frac{3}{\tilde{V}^2+\tilde{V}+1}}\dfrac{\partial \varepsilon}{\partial h}}.$$

Now, it hold that

$$\frac{d\mathcal{E}(h,s(h;\tilde{V}))}{dh} = \sqrt{\frac{3}{\tilde{V}^2+\tilde{V}+1}} \cdot \frac{\partial \varepsilon}{\partial h},$$

$$\frac{d\mathcal{M}(h,s(h;\tilde{V}))}{dh} = \mathcal{M}_h + \mathcal{M}_s \cdot \frac{ds(h;\tilde{V})}{dh}$$

and

$$\mathcal{A}_h + \mathcal{A}_s \cdot \frac{ds(h;\tilde{V})}{dh} = 0.$$

Therefore,

$$\frac{\partial \boldsymbol{m}(\tilde{V},\varepsilon^2)}{\partial \varepsilon} = -\frac{\mathcal{M}_s \cdot \mathcal{A}_h - \mathcal{M}_h \cdot \mathcal{A}_s}{\mathcal{A}_s \cdot \dfrac{d\mathcal{E}(h,s(h;\tilde{V}))}{dh}}. \tag{6.9}$$

Thus, we get (6.1).

We have (6.7) by

$$\mathcal{D}^{K2}(h,s) = \left(3s^4 - 4s^3 + 4s^2\right)\left(h - \frac{2s^2 - s + 2}{s\,(3s^2 - 4s + 4)}\right)^2 + \frac{8\left(s^2 - s + 1\right)(s-1)^2}{3s^2 - 4s + 4} > 0.$$

We have (6.8) by Lemmas 4.4 and 4.6. Thus, we complete the proof. □

Proposition 6.2 *Let $\tilde{V} > 0$ be fixed. Then, the following equation holds:*

$$\mathcal{M}_s \cdot \mathcal{A}_h - \mathcal{M}_h \cdot \mathcal{A}_s = \frac{32s(1-s)^2(1-hs)^2}{h\,(1-h)\,(\mathcal{D}^{K2})^3 \cdot K(\sqrt{h})^2} \cdot \frac{\sqrt{s}\cdot \mathcal{N}^{\Pi}}{\sqrt{(1-s)(1-sh)}} \cdot \mathcal{J}, \tag{6.10}$$

where

$$\mathcal{N}^{\Pi}(h,s) := \Big(2s^4h^4 - \left(2s^4+4s^3\right)h^3 + \left(2s^4-4s^3+12s^2-4s+2\right)h^2 - (4s+2)\,h + 2\Big)E(\sqrt{h}) - \Big(s^4h^4 + \left(-3s^4+4s^3-6s^2\right)h^3 + \left(2s^4-4s^3+6s^2+4s+1\right)h^2 + (-4s-3)\,h + 2\Big)K(\sqrt{h}), \tag{6.11}$$

$$\mathcal{J}(h,s) := -\frac{\sqrt{(1-s)(1-sh)}}{\sqrt{s}}\cdot\Pi(-hs,\sqrt{h}) + \frac{\sqrt{(1-s)(1-sh)}}{\sqrt{s}}\cdot\frac{\mathcal{N}^{KE}K(\sqrt{h})E(\sqrt{h}) + \mathcal{N}^{K2}K(\sqrt{h})^2}{\mathcal{N}^{\Pi}}, \tag{6.12}$$

$$\mathcal{N}^{KE}(h,s) := -h^3s^3 - h^2s^3 + 6h^2s^2 - 3h^2s + 2h^2 - 3hs - 2h + 2, \tag{6.13}$$

$$\mathcal{N}^{K2}(h,s) := (1-h)\left(h^2s^3 - 3h^2s^2 + 3hs + h - 2\right). \tag{6.14}$$

Moreover, it holds that

$$\mathcal{N}^{\Pi}(h,s) > 0 \quad (0<h<1,\quad 0<s<1). \tag{6.15}$$

Figure 6.1 shows graphs of $\mathcal{N}^{\Pi}(h,s)$ ($0<h<1$, $0<s<1$).

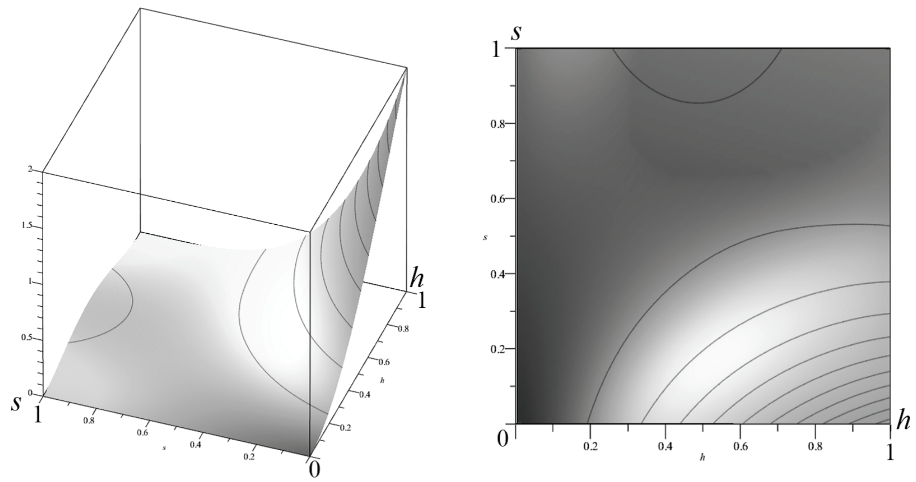

Figure 6.1: Graphs of $\mathcal{N}^{\Pi}(h,s)$

Proof. We have (6.2) - (6.5). Multiplying $\mathcal{M}_s$ by $\mathcal{A}_h$. we get

$$\begin{aligned}
&\mathcal{M}_s \cdot \mathcal{A}_h \\
&= 32\, s\,(1-s)^2\,(1-hs)\left(h^2 s^3 - 2\,hs^3 + 3\,hs^2 - 3\,hs + 2\,h - 1\right) \\
&\cdot \Big(3\,\left(hs^2-1\right)\left(hs^2 - 2\,hs + 1\right)\left(hs^2 - 2\,s + 1\right)\Pi(-hs,\sqrt{h}) \\
&+ s\mathcal{D}^{K2} E(\sqrt{h}) \\
&+ \left(hs^2 - 2\,s + 1\right)\left(2\,h^2 s^3 - hs^3 - hs^2 - 4\,hs + 2\,s^2 - s + 3\right) K(\sqrt{h})\Big) \\
&\cdot \left(\mathcal{D}^{K2}\right)^{-4} K(\sqrt{h})^{-1}.
\end{aligned}$$

Multiplying $\mathcal{M}_h$ by $\mathcal{A}_s$. we get

$$\begin{aligned}
&\mathcal{M}_h \cdot \mathcal{A}_s \\
&= -4\,(1-s)\left(hs^2 - 2\,hs + 1\right)\left(hs^2 - 2\,s + 1\right)\left(1 - hs^2\right) \\
&\cdot \Big(\Big((1-hs)\,\mathcal{D}^{K2} E(\sqrt{h}) \\
&- (1-h)\left(h^2 s^4 + 2h^2 s^3 - 8hs^3 + 8hs^2 - 6hs + 4s^2 - 4s + 3\right) K(\sqrt{h})\Big) \\
&\cdot \Pi(-hs,\sqrt{h}) - \mathcal{D}^{K2} E(\sqrt{h}) K(\sqrt{h}) \\
&+ (1-h)\left(h^2 s^4 - 4\,hs^3 + 4\,hs^2 - 4\,hs + 4\,s^2 - 4\,s + 3\right) K(\sqrt{h})^2\Big) \\
&\cdot (h(1-h))^{-1}\left(\mathcal{D}^{K2}\right)^{-3} K(\sqrt{h})^{-2}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\mathcal{M}_s \cdot \mathcal{A}_h - \mathcal{M}_h \cdot \mathcal{A}_s \\
&= \frac{32s(1-s)^2(1-hs)^2}{h\,(1-h)\,(\mathcal{D}^{K2})^3 \cdot K(\sqrt{h})^2} \\
&\cdot \left(-\mathcal{N}^{\Pi} \cdot \Pi(-hs,\sqrt{h}) + \mathcal{N}^{KE} \cdot K(\sqrt{h})E(\sqrt{h}) + \mathcal{N}^{K2} K(\sqrt{h})^2\right) \qquad (6.16)
\end{aligned}$$

by direct calculation. Thus (6.10) is obvious from (6.16).

We show (6.15). We see from the proof of Lemma 4.6 that

$$2f(h,s)E(\sqrt{h}) - P(h,s)K(\sqrt{h}) > 0 \quad (0 < h < 1)$$

and

$$\begin{aligned}
\mathcal{N}^{\Pi}(h,s) &= 2f(h,s)E(\sqrt{h}) - P(h,s)K(\sqrt{h}) \\
&= K(\sqrt{h})\left(2f(h,s)\frac{E(\sqrt{h})}{K(\sqrt{h})} - P(h,s)\right) \\
&> K(\sqrt{h})(2f(h,s)\sqrt{1-h} - P(h,s)) \\
&> 0 \quad (0 < h < 1,\ 0 < s < 1),
\end{aligned}$$

where $f(h,s)$ and $P(h,s)$ be defined by (4.35) and (4.49). Thus we complete the proof. $\square$

We note that $(\tilde{V},\varepsilon^2)$ with $0<\tilde{V}<1$ and $0<\varepsilon^2<\tilde{V}/\pi^2$ corresponds to (h,s) with $0<h<1$ and $0<s<1/(1+\sqrt{1-h})$ by (4.42).

Now we investigate $\mathcal{J}(h,s)$. Figure 6.2 shows graphs of $\mathcal{J}(h,s)$ for $(0<h<1,\ 0<s<1/(1+\sqrt{1-h}))$

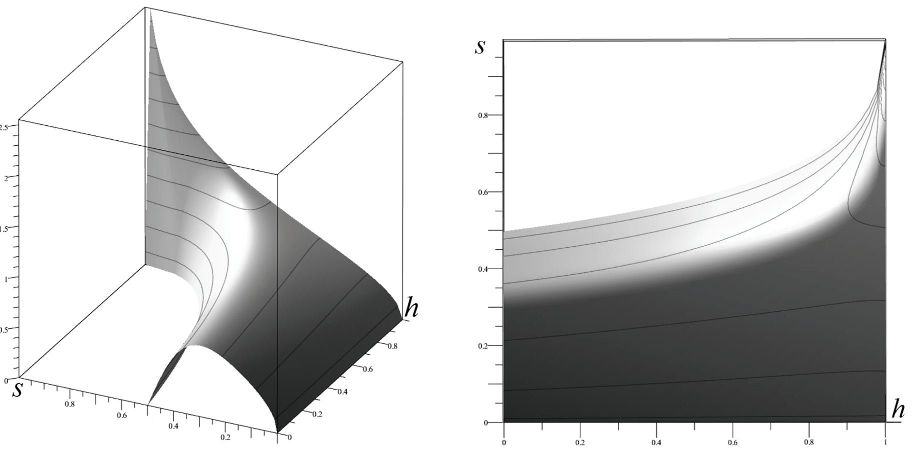

Figure 6.2: Graphs of $\mathcal{J}(h,u)$

The following propositions are crucial for the proof of Theorem 3.1. We will give a proof of it in Subsection 6.2 and 6.3.

Proposition 6.3 *Let $\mathcal{J}(h,s)$ be defined by (6.12) For each fixed $h\in(0,1)$, it holds that*

$$\lim_{s\downarrow 0}\mathcal{J}(h,s)=0, \tag{6.17}$$

$$\mathcal{J}\left(h,\frac{1}{1+\sqrt{1-h}}\right)=0 \tag{6.18}$$

for each fixed $h\in(0,1)$.

Proposition 6.4 *Let $\mathcal{J}(h,s)$ be defined by (6.12). Then $\mathcal{J}(h,s)$ satisfies the following equation*

$$\frac{\partial}{\partial s}\mathcal{J}(h,s)=\frac{F(h,s)K(\sqrt{h})^3}{\sqrt{s(1-s)(1-sh)}\cdot(\mathcal{N}^{\Pi})^2}, \tag{6.19}$$

where

$$
\begin{aligned}
F(h,s) := {} & h^4\mathcal{C}^{F0}(h,U(h))s^8 - 2h^4\mathcal{C}^{F1}(h,U(h))s^7 + 2h^4\mathcal{C}^{F2}(h,U(h))s^6 \\
& - 2h^3\mathcal{C}^{F3}(h,U(h))s^5 + 2h^2\mathcal{C}^{F4}(h,U(h))s^4 - 2h^2\mathcal{C}^{F3}(h,U(h))s^3 \\
& + 2h^2\mathcal{C}^{F2}(h,U(h))s^2 - 2h\mathcal{C}^{F1}(h,U(h))s + \mathcal{C}^{F0}(h,U(h)),
\end{aligned} \tag{6.20}
$$

$$
\begin{aligned}
\mathcal{C}^{F0}(h,u) := {} & 2\left(h^2-h+1\right)^2 u^3 - 3\,(2-h)\,(1-h)\left(h^2-h+1\right)u^2 \\
& + 3\left(h^2-2\,h+2\right)(1-h)^2\,u - (2-h)\,(1-h)^3\,,
\end{aligned} \tag{6.21}
$$

$$
\begin{aligned}
\mathcal{C}^{F1}(h,u) := {} & 4\,(h+1)\left(h^2-h+1\right)u^3 - 3\,(1-h)\left(3\,h^2-3\,h+4\right)u^2 \\
& + 6\left(h^2-2\,h+2\right)(1-h)^2\,u - (4-3\,h)\,(1-h)^3\,,
\end{aligned} \tag{6.22}
$$

$$
\begin{aligned}
\mathcal{C}^{F2}(h,u) := {} & 4\left(4\,h^2-h+4\right)u^3 - (1-h)\left(23\,h^2-6\,h+25\right)u^2 \\
& + 2\left(3\,h^2-4\,h+4\right)(1-h)^2\,u + (3\,h+1)\,(1-h)^3\,,
\end{aligned} \tag{6.23}
$$

$$
\begin{aligned}
\mathcal{C}^{F3}(h,u) := {} & 4\,(h{+}1)\left(h^2+5h+1\right)u - (1-h)\left(16h^3+31\,h^2+41h-4\right)u^2 \\
& + 2\left(h^2+6\,h-10\right)(1-h)^2\,u + (5\,h+12)\,(1-h)^3\,,
\end{aligned} \tag{6.24}
$$

$$
\begin{aligned}
\mathcal{C}^{F4}(h,u) := {} & 2\left(h^4+2\,h^3+29\,h^2+2\,h+1\right)u^3 \\
& - (1-h)\left(8\,h^4+17\,h^3+87\,h^2-5\,h-2\right)u^2 \\
& - \left(8h^3-33h^2+30h+10\right)(1-h)^2\,u - \left(2h^2-21h-6\right)(1-h)^3\,,
\end{aligned} \tag{6.25}
$$

$$
U(h) := \frac{E(\sqrt{h})}{K(\sqrt{h})}. \tag{6.26}
$$

Let us consider properties of $F(h,s)$. Figure 6.3 shows a graphs of $F(h,s)$ for $(0 < h < 1,\ 0 < s < 1/(1+\sqrt{1-h}))$

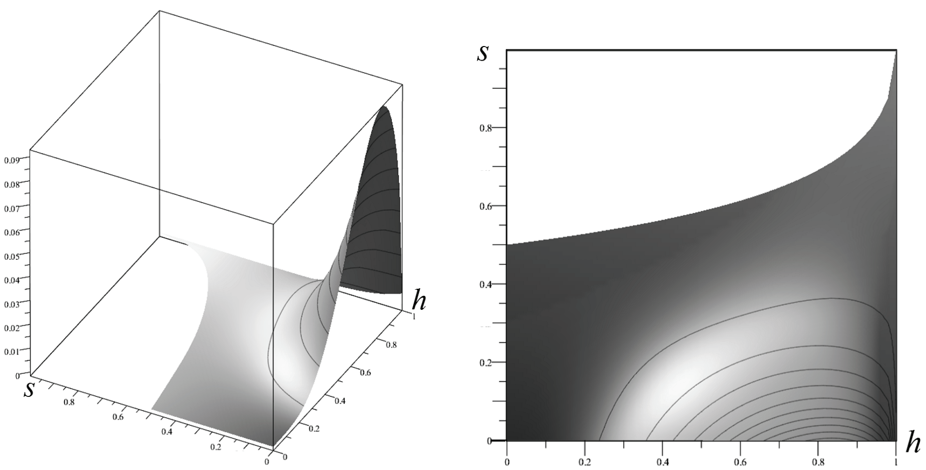

Figure 6.3: Graphs of $F(h,s)$

Proposition 6.5 *Let $h \in (0,1)$ be fixed. Then,*

$$F(h,0) = C^{F0}\left(h, \frac{E(\sqrt{h})}{K(\sqrt{h})}\right) > 0, \tag{6.27}$$

$$F\left(h, \frac{1}{1+\sqrt{1-h}}\right) = -\frac{8(1-h)^4}{\left(1+(1-h)^{1/2}\right)^4} \cdot g_1\left(\sqrt{1-h}, \frac{E(\sqrt{h})}{K(\sqrt{h})}\right) < 0, \tag{6.28}$$

where $F(h,s)$ is defined by (6.20),

$$g_1(H,u) := \left((1+H^2)u - 2H^2\right)\left(-(1+H^2)u^2 + 2(1+H^4)u - (1+H^2)H^2\right). \tag{6.29}$$

The following propositions are crucial for the proof of Theorem 3.1. We will give a proof of it in Section 7

Proposition 6.6 *Let $h \in (0,1)$ be fixed. Then, the following equation*

$$F(h,s) = 0 \quad \left(0 < s < \frac{1}{1+\sqrt{1-h}}\right) \tag{6.30}$$

in s has the unique solution, where $F(h,s)$ is defined by (6.20).

Proof of Theorem 3.1. We have

$$\mathcal{J}(h,s) > 0 \quad \left(0 < h < 1, \quad 0 < s < \frac{1}{1+\sqrt{1-h}}\right) \tag{6.31}$$

by Propositions 6.3-6.6. Hence, we obtain (3.4) by Propositions 6.1 and 6.2.

We get (3.5) by (1.14). We obtain (3.6) by (3.4) and (1.13).

6.2 Proofs of Propositions 6.3-6.4

We prepare two lemmas to prove Proposition 6.3.

Lemma 6.1 *Let $h \in (0,\ 1)$ be fixed. $\mathcal{J}(h,s)$ defined by (6.12) satisfies (6.17).*

Proof. Let $h \in (0,\ 1)$ be fixed. We have

$$\begin{aligned} &\mathcal{J}(h,s)\cdot\sqrt{s} \\ &= -\sqrt{(1-s)(1-sh)}\cdot\Pi(-hs,\sqrt{h}) \\ &+ \sqrt{(1-s)(1-sh)}\cdot K(\sqrt{h})\cdot\frac{\mathcal{N}^{KE}(h,s)E(\sqrt{h}) + \mathcal{N}^{K2}(h,s)K(\sqrt{h})}{\mathcal{N}^{\Pi}(h,s)} = (*). \end{aligned}$$

We note that

$$\begin{aligned}\mathcal{N}^{\Pi}(h,0) &= 2\,\left(h^2-h+1\right)E(\sqrt{h})-(1-h)\,(2-h)\,K(\sqrt{h})\\ &\geq \left(2\,\left(h^2-h+1\right)\sqrt{1-h}-(1-h)\,(2-h)\right)K(\sqrt{h})\\ &= \sqrt{1-h}\left(4-2h+3\sqrt{1-h}\right)\left(\sqrt{1-h}-1\right)^2 K(\sqrt{h})>0\quad(0<h<1).\end{aligned}$$

Hence we get

$$\begin{aligned}&(*)|_{s=0}\\ &= -\Pi(0,\sqrt{h})+K(\sqrt{h})\cdot\frac{\mathcal{N}^{KE}(h,0)E(\sqrt{h})+\mathcal{N}^{K2}(h,0)K(\sqrt{h})}{\mathcal{N}^{\Pi}(h,0)}\\ &= -K(\sqrt{h})+K(\sqrt{h})\cdot\frac{2\,\left(h^2-h+1\right)E(\sqrt{h})-(1-h)\,(2-h)\,K(\sqrt{h})}{2\,(h^2-h+1)\,E(\sqrt{h})-(1-h)\,(2-h)\,K(\sqrt{h})}=0.\end{aligned}$$

Thus we obtain

$$(*) = c_1^{\mathcal{J}}(h)\cdot s + c_2^{\mathcal{J}}(h)\cdot s^2+\cdots,$$

near $s=0$, where $c_i^{\mathcal{J}}(h)$, $(i=1,2,\cdots)$ are some constants. Therefore we have

$$\mathcal{J}(h,s) = c_1^{\mathcal{J}}(h)\cdot\sqrt{s}+c_2^{\mathcal{J}}(h)\cdot s\sqrt{s}+\cdots,$$

which implies (6.17).

Lemma 6.2 *Let $h\in(0,1)$ be fixed. $\mathcal{J}(h,s)$ defined by (6.12) satisfies (6.18).*

Proof. We have

$$\mathcal{J}(h,s) = -Q(h,s)\cdot\frac{\partial \boldsymbol{m}(\tilde{V},\varepsilon^2)}{\partial\varepsilon},$$

where

$$Q(h,s) := \mathcal{A}_s\cdot\frac{d\mathcal{E}(h,s(h;\tilde{V}))}{dh}\cdot\frac{h\,(1-h)\left(\mathcal{D}^{K2}\right)^3\cdot K(\sqrt{h})^2}{32s(1-s)^2(1-hs)^2}\cdot\frac{\sqrt{(1-s)(1-sh)}}{\sqrt{s}\cdot\mathcal{N}^{\Pi}}$$

by (6.10) and (6.1). Thus

$$\mathcal{J}\left(h,\frac{1}{1+\sqrt{1-h}}\right) = -Q\left(h,\frac{1}{1+\sqrt{1-h}}\right)\cdot\frac{\partial\boldsymbol{m}(1,\varepsilon^2)}{\partial\varepsilon}=0$$

by (1.14).

Proof of Proposition 6.3. It is obvious from Lemmas 6.1 and 6.2. □
Proof of Proposition 6.4. We have

$$\frac{\partial}{\partial s}\left(-\frac{\sqrt{(1-s)(1-sh)}}{\sqrt{s}}\Pi(-hs,\sqrt{h})\right) = \frac{sE(\sqrt{h})+(1-s)K(\sqrt{h})}{2s\sqrt{s(1-s)(1-sh)}}, \quad (6.32)$$

$$\frac{\partial}{\partial s}\left(\frac{\sqrt{(1-s)(1-sh)}}{\sqrt{s}}\right) = \frac{hs^2-1}{2s\sqrt{s(1-s)(1-sh)}}, \quad (6.33)$$

$$\begin{aligned}\frac{\partial}{\partial s}\mathcal{N}^{\Pi} &= 4h\Big(\left(2h^3s^3-2h^2s^3-3h^2s^2+2hs^3-3hs^2+6hs-h-1\right)E(\sqrt{h})\\ &\quad +(1-h)\left(h^2s^3-2\,hs^3+3\,hs^2-3\,hs+1\right)K(\sqrt{h})\Big), \quad (6.34)\end{aligned}$$

$$\begin{aligned}&\frac{\partial}{\partial s}\left(\mathcal{N}^{KE}K(\sqrt{h})E(\sqrt{h})+\mathcal{N}^{K2}K(\sqrt{h})^2\right)\\ &= -3hK(\sqrt{h})\Big(\left(h^2s^2+s^2h-4sh+h+1\right)E(\sqrt{h})\\ &\quad -(1-h)\left(s^2h-2sh+1\right)K(\sqrt{h})\Big). \quad (6.35)\end{aligned}$$

Hence, we obtain (6.19) by direct calculation. Thus, we complete proof. □

6.3 Proof of Proposition 6.5

We begin with the following lemmas.

Lemma 6.3 *Let $F(h,u)$ and $C^{F0}(h,u)$ be defined by (6.20) and (6.21) respectively, then (6.27) holds.*

Proof. We have

$$\begin{aligned}&F(h,0) = C^{F0}\left(h,\frac{E(\sqrt{h})}{K(\sqrt{h})}\right),\\ &C^{F0}(h,H) = H^3\left(H^2+H+1\right)\left(2\,H^2+3\,H+2\right)(1-H)^4 > 0,\\ &C^{F0}_u(h,u)\\ &= 6\left(h^2-h+1\right)^2u^2-6\,(2-h)\,(1-h)\left(h^2-h+1\right)u+3\left(h^2-2h+2\right)(1-h)^2\\ &= 6\left(h^2-h+1\right)^2\left(u-\frac{1}{2}\cdot\frac{(2-h)\,(1-h)}{h^2-h+1}\right)^2+\frac{3}{2}\,(1-h)^2\,h^2 > 0\end{aligned}$$

for $0<h<1$, $H<u<1$, where $H=\sqrt{1-h}$. Thus, we complete the proof by using $\sqrt{1-h}<E(\sqrt{h})/K(\sqrt{h})<1$ due to Lemma 2.1. □

Lemma 6.4 *Let $F(h,u)$ and $g_1(H,u)$ be defined by (6.20) and (6.29) respectively, then (6.28) holds.*

Proof. We have

$$F\left(h,\frac{1}{1+\sqrt{1-h}}\right)=F\left(h,\frac{1}{1+H}\right)=-\frac{8H^8}{(1+H)^4}\cdot g_1\left(H,\frac{E(\sqrt{h})}{K(\sqrt{h})}\right), \tag{6.36}$$

where $H=\sqrt{1-h}$. Hence we may show that

$$g_1(H,u)>0 \quad (0<H<1, \quad H<u<1) \tag{6.37}$$

by Lemma 2.1.

It is easy to see that

$$\left((1+H^2)u-2H^2\right)\geq H(1-H^2)>0 \ \ (0<H<1,\ H<u<1)$$

and

$$\begin{aligned}&-(1+H^2)u^2+2(1+H^4)u-(1+H^2)H^2\\&\geq \min\left\{2H(1-H)^2(H^2+H+1),\ (1-H^2)^2\right\}>0 \ \ (0<H<1,\ H<u<1).\end{aligned}$$

Thus we have (6.37). Therefore we complete the proof. □

Thus we obtain Proposition 6.5 by Lemmas 6.3 and 6.4.

7 Proof of Proposition 6.6

We prepare several lemmas to prove Proposition 6.6.

7.1 Key lemmas and proof of Proposition 6.6

Lemma 7.1 *Let $F(h,u)$ be defined by (6.20). It holds that*

$$\frac{\partial F}{\partial s}(h,s) = 8h^4\mathcal{C}^{F0}s^7 - 14h^4\mathcal{C}^{F1}s^6 + 12h^4\mathcal{C}^{F2}s^5 - 10h^3\mathcal{C}^{F3}s^4 + 8h^2\mathcal{C}^{F4}s^3 - 6h^2\mathcal{C}^{F3}s^2 + 4h^2\mathcal{C}^{F2}s - 2h\mathcal{C}^{F1}, \tag{7.1}$$

$$\frac{\partial^2 F}{\partial s^2}(h,s) = 56h^4\mathcal{C}^{F0}s^6 - 84h^4\mathcal{C}^{F1}s^5 + 60h^4\mathcal{C}^{F2}s^4 - 40h^3\mathcal{C}^{F3}s^3 + 24h^2\mathcal{C}^{F4}s^2 - 12h^2\mathcal{C}^{F3}s + 4h^2\mathcal{C}^{F2}, \tag{7.2}$$

$$\frac{\partial^3 F}{\partial s^3}(h,s) = 336h^4\mathcal{C}^{F0}s^5 - 420h^4\mathcal{C}^{F1}s^4 + 240h^4\mathcal{C}^{F2}s^3 - 120h^3\mathcal{C}^{F3}s^2 + 48h^2\mathcal{C}^{F4}s - 12h^2\mathcal{C}^{F3}, \tag{7.3}$$

$$\frac{\partial^4 F}{\partial s^4}(h,s) = 1680h^4\mathcal{C}^{F0}s^4 - 1680h^4\mathcal{C}^{F1}s^3 + 720h^4\mathcal{C}^{F2}s^2 - 240h^3\mathcal{C}^{F3}s + 48h^2\mathcal{C}^{F4}, \tag{7.4}$$

$$\frac{\partial^5 F}{\partial s^5}(h,s) = 6720h^4\mathcal{C}^{F0}s^3 - 5040h^4\mathcal{C}^{F1}s^2 + 1440h^4\mathcal{C}^{F2}s - 240h^3\mathcal{C}^{F3}, \tag{7.5}$$

$$\frac{\partial^6 F}{\partial s^6}(h,s) = 20160h^4\mathcal{C}^{F0}s^2 - 10080h^4\mathcal{C}^{F1}s + 1440h^4\mathcal{C}^{F2}, \tag{7.6}$$

$$\frac{\partial^7 F}{\partial s^7}(h,s) = 40320h^4\mathcal{C}^{F0}s - 10080h^4\mathcal{C}^{F1}, \tag{7.7}$$

$$\frac{\partial^8 F}{\partial s^8}(h,s) = 40320h^4\mathcal{C}^{F0} > 0 \quad (0 < h < 1)\,, \tag{7.8}$$

where

$$C^{F0} = C^{F0}(h,U(h))\,,\ C^{F1} = C^{F1}(h,U(h))\,,\ C^{F2} = C^{F2}(h,U(h))\,,$$
$$C^{F3} = C^{F3}(h,U(h))\,,\ C^{F4} = C^{F4}(h,U(h))\,,\ U(h) = E(\sqrt{h})/E(\sqrt{h})$$

are defined by (6.21)-(6.26).

Proof. We obtain identities in (7.1)-(7.8) by direct calculations and the inequality in (7.8) by Lemma 6.3. □

Lemma 7.2 *Let $F(h,u)$ be defined by (6.20). It holds that*

$$\frac{\partial F}{\partial s}\left(h, \frac{1}{1+\sqrt{1-h}}\right) = \frac{\partial F}{\partial s}\left(1-H^2, \frac{1}{1+H}\right)$$
$$= \frac{32H^7(1-H)}{(1+H)^3}\cdot g_1\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) > 0 \quad (0<h<1), \tag{7.9}$$

$$\frac{\partial^2 F}{\partial s^2}\left(h, \frac{1}{1+\sqrt{1-h}}\right) = \frac{\partial^2 F}{\partial s^2}\left(1-H^2, \frac{1}{1+H}\right)$$
$$= -\frac{16H^6(1-H)^2}{(1+H)^2}\cdot g_2\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) < 0 \quad \left(1-\frac{31^2}{100^2} \le h < 1\right), \tag{7.10}$$

$$\frac{\partial^3 F}{\partial s^3}\left(h, \frac{1}{1+\sqrt{1-h}}\right) = \frac{\partial^3 F}{\partial s^3}\left(1-H^2, \frac{1}{1+H}\right)$$
$$= \frac{48H^5(1-H)^3}{1+H}\cdot g_3\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) \begin{cases} <0 & (0<h<h_0), \\ =0 & (h=h_0), \\ >0 & (h_0<h<1), \end{cases} \tag{7.11}$$

$$\frac{\partial^4 F}{\partial s^4}\left(h, \frac{1}{1+\sqrt{1-h}}\right) = \frac{\partial^4 F}{\partial s^4}\left(1-H^2, \frac{1}{1+H}\right)$$
$$= 48H^2(1-H)^2\cdot g_4\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) > 0 \quad \left(0<h<1-\frac{1}{10^2}\right), \tag{7.12}$$

$$\frac{\partial^5 F}{\partial s^5}\left(h, \frac{1}{1+\sqrt{1-h}}\right) = \frac{\partial^5 F}{\partial s^5}\left(1-H^2, \frac{1}{1+H}\right)$$
$$= -480H^3(1+H)(1-H)^3\cdot g_5\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) < 0 \quad (0<h<1), \tag{7.13}$$

$$\frac{\partial^6 F}{\partial s^6}\left(h, \frac{1}{1+\sqrt{1-h}}\right) = \frac{\partial^6 F}{\partial s^6}\left(1-H^2, \frac{1}{1+H}\right)$$
$$= 1440H^2(1+H)^2(1-H)^4\cdot g_6\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) > 0 \quad (0<h<1), \tag{7.14}$$

$$\frac{\partial^7 F}{\partial s^7}\left(h, \frac{1}{1+\sqrt{1-h}}\right) = \frac{\partial^7 F}{\partial s^7}\left(1-H^2, \frac{1}{1+H}\right)$$
$$= -10080H(1+H)^3(1-H)^5\cdot g_7\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) < 0 \quad (0<h<1), \tag{7.15}$$

where $H := \sqrt{1-h}$,

$$\begin{aligned} g_2(H,u) &\\ := -2\left(H^2+1\right)&\left(5\,H^2+3\,H+5\right)u^3\\ &-\left(-12H^6+4H^5-37H^4-22H^3-37H^2+4H-12\right)u^2\\ &-2\,H^2\left(18H^4+H^3+8H^2+H+18\right)u+H^4\left(15\,H^2+2\,H+15\right), \end{aligned} \tag{7.16}$$

$$\begin{aligned} &g_3(H,u)\\ &:= -2\left(H^2+1\right)\left(8\,H^2+9\,H+8\right)u^3\\ &\quad+\left(8\,H^6-12\,H^5+55\,H^4+66\,H^3+55\,H^2-12\,H+8\right)u^2\\ &\quad-2H^2\left(19H^4+3H^3+10H^2+3H+19\right)u+H^4\left(17H^2+6H+17\right), \end{aligned} \tag{7.17}$$

$$\begin{aligned} &g_4(H,u)\\ &:= \left(72\,H^6-16\,H^5-104\,H^3-16\,H+72\right)u^3\\ &\quad-\left(8H^8-64H^7+329H^6-148H^5-258H^4-148H^3+329H^2-64H+8\right)u^2\\ &\quad+2\,H^2\left(49\,H^6-92\,H^5+51\,H^4-36\,H^3+51\,H^2-92\,H+49\right)u\\ &\quad-H^4\left(47\,H^4-76\,H^3+34\,H^2-76\,H+47\right), \end{aligned} \tag{7.18}$$

$$\begin{aligned} &g_5(H,u)\\ &:= -\left(-28\,H^6-12\,H^5+12\,H^4+48\,H^3+12\,H^2-12\,H-28\right)u^3\\ &\quad+H\left(8\,H^6-95\,H^5+36\,H^4+110\,H^3+36\,H^2-95\,H+8\right)u^2\\ &\quad+2\,H^2\left(9\,H^6-22\,H^5+7\,H^4-8\,H^3+7\,H^2-22\,H+9\right)u\\ &\quad-H^4\left(9\,H^4-20\,H^3-2\,H^2-20\,H+9\right), \end{aligned} \tag{7.19}$$

$$\begin{aligned} &g_6(H,u)\\ &:= \left(28\,H^6+28\,H^5-12\,H^4-52\,H^3-12\,H^2+28\,H+28\right)u^3\\ &\quad-H^2\left(65\,H^4-17\,H^3-80\,H^2-17\,H+65\right)u^2\\ &\quad+2\,H^2\left(3\,H^6-15\,H^5+H^4-4\,H^3+H^2-15\,H+3\right)u\\ &\quad-H^4\left(3\,H^4-15\,H^3-8\,H^2-15\,H+3\right), \end{aligned} \tag{7.20}$$

$$\begin{aligned} &g_7(H,u)\\ &:= 4\left(2\,H^2+3\,H+2\right)\left(H^4-H^2+1\right)u^3\\ &\quad-3H^2\left(4H^4+H^3-2H^2+H+4\right)u^2\\ &\quad-6H^3\left(H^4+1\right)u+H^5\left(3H^2+2H+3\right). \end{aligned} \tag{7.21}$$

Here, h_0 is a real number with $h \in (1 - 31^2/100^2,\ 1 - 1/10^2)$ which appears in Lemma 7.9.

Proof. We get identities in (7.9)-(7.15) by direct calculations. We have inequality in (7.9) by Lemma 6.4, We obtain (7.10), (7.11), (7.12), (7.13), (7.14), (7.15) by Lemma 7.7, Lemma 7.9, Lemma 7.10, Lemma 7.11, Lemma 7.12, Lemma 7.13 which we prove in Subsection 7.2. □

Table 7.1 shows the sign of $F(0,h)$, $\cdots$, $\partial^8 F(0,h)/\partial s^8$ and $F(1/(1+\sqrt{1-h}),h)$, $\cdots$, $\partial^8 F(1/(1+\sqrt{1-h}),h)/\partial s^8$, which we will prove in subsequent lemmas 7.3-7.6. Here $*$ means $+$, $-$ or 0.

	$s=0$	$s=\dfrac{1}{1+\sqrt{1-h}}$		
	$0<h<1$	$0<h<h_0$	$h=h_0$	$h_0<h<1$
F	$+$	$-$	$-$	$-$
$\dfrac{\partial F}{\partial s}$	$-$	$+$	$+$	$+$
$\dfrac{\partial^2 F}{\partial s^2}$	$+$	$*$	$-$	$-$
$\dfrac{\partial^3 F}{\partial s^3}$	$-$	$-$	0	$+$
$\dfrac{\partial^4 F}{\partial s^4}$	$+$	$+$	$+$	$*$
$\dfrac{\partial^5 F}{\partial s^5}$	$-$	$-$	$-$	$-$
$\dfrac{\partial^6 F}{\partial s^6}$	$+$	$+$	$+$	$+$
$\dfrac{\partial^7 F}{\partial s^7}$	$-$	$-$	$-$	$-$
$\dfrac{\partial^8 F}{\partial s^8}$	$+$	$+$	$+$	$+$

Table 7.1: Signs of F, $\cdots$, $\partial^8 F/\partial s^8$ at $s=0$ and $s=1/(1+\sqrt{1-h})$

Lemma 7.3 *Let $F(h,u)$ be defined by (6.20). It holds that*

$$\frac{\partial F}{\partial s}(h,0) = \frac{1}{5040h^3}\cdot\frac{\partial^7 F}{\partial s^7}(h,0) < 0 \quad (0 < h < 1), \tag{7.22}$$

$$\frac{\partial^2 F}{\partial s^2}(h,0) = \frac{1}{360h^3}\cdot\frac{\partial^6 F}{\partial s^6}(h,0) > 0 \quad (0 < h < 1), \tag{7.23}$$

$$\frac{\partial^3 F}{\partial s^3}(h,0) = \frac{1}{20h^2}\cdot\frac{\partial^5 F}{\partial s^5}(h,0) < 0 \quad (0 < h < 1), \tag{7.24}$$

$$\frac{\partial^5 F}{\partial s^5}(h,s) < 0 \quad \left(0 \le h < 1,\ \ 0 \le s \le \frac{1}{1+\sqrt{1-h}}\right), \tag{7.25}$$

$$\frac{\partial^6 F}{\partial s^6}(h,s) > 0 \quad \left(0 \le h < 1,\ \ 0 \le s \le \frac{1}{1+\sqrt{1-h}}\right), \tag{7.26}$$

$$\frac{\partial^7 F}{\partial s^7}(h,s) < 0 \quad \left(0 \le h < 1,\ \ 0 \le s \le \frac{1}{1+\sqrt{1-h}}\right). \tag{7.27}$$

Proof. We obtain (7.27) by (7.8) and (7.15). Hence we get (7.26) by (7.27) and (7.14). Therefore, we obtain (7.25) by (7.26) and (7.13). Thus we obtain equalities in (7.22), (7.23), (7.24) by Lemma 7.1, and inequalities by (7.27), (7.26), (7.25) with $s = 0$. □

Lemma 7.4 *Let $F(h,u)$ be defined by (6.20). It holds that*

$$\frac{\partial^2 F}{\partial s^2}\left(h_0, \frac{1}{1+\sqrt{1-h_0}}\right) < 0, \tag{7.28}$$

$$\frac{\partial^4 F}{\partial s^4}\left(h, \frac{1}{1+\sqrt{1-h}}\right) > 0 \quad (0 < h \le h_0). \tag{7.29}$$

Proof. We obtain (7.28) by (7.10) and $h_0 \in (1 - 31^2/100^2,\ 1)$. We get (7.29) by (7.12) and $h_0 \in (0,\ 1 - 1/10^2)$. Thus we complete the proof. □

Lemma 7.5 *Let $h \in (0,\ h_0]$ be fixed. Then, (6.30) in s has the unique solution.*

Proof. Let us fixed $h \in (0,\ h_0]$. We get

$$\frac{\partial^4 F}{\partial s^4}(h,s) > 0 \quad \left(0 \le s \le \frac{1}{1+\sqrt{1-h}}\right) \tag{7.30}$$

by (7.29) and (7.25). Hence we obtain

$$\frac{\partial^3 F}{\partial s^3}(h,s) \le 0 \quad \left(0 \le s \le \frac{1}{1+\sqrt{1-h}}\right) \tag{7.31}$$

by (7.30) and (7.11) with $0 < h \le h_0$.

On the other hand, we have

$$\frac{\partial F}{\partial s}(h,0)<0, \quad \frac{\partial F}{\partial s}\left(h, \frac{1}{1+\sqrt{1-h}}\right)>0 \tag{7.32}$$

by (7.22) and (7.9), respectively. Therefore there exists the unique $s_1(h) \in (0, 1/(1+\sqrt{1-h}))$ such that

$$\frac{\partial F}{\partial s}(h,s)<0 \quad (0 \leq s < s_1(h)), \tag{7.33}$$

$$\frac{\partial F}{\partial s}(h,s_1(h))=0, \tag{7.34}$$

$$\frac{\partial F}{\partial s}(h,s)<0 \quad \left(s_1(h) < s \leq \frac{1}{1+\sqrt{1-h}}\right) \tag{7.35}$$

by (7.31).

Now, we have

$$F(h,0)>0, \quad F\left(h, \frac{1}{1+\sqrt{1-h}}\right)<0 \tag{7.36}$$

due to (6.27) and (6.28). Consequently, there exists the unique $s_0(h) \in (0, 1/(1+\sqrt{1-h}))$ such that

$$F(h,s)>0 \quad (0 \leq s < s_0(h)), \tag{7.37}$$

$$F(h,s_0(h))=0, \tag{7.38}$$

$$F(h,s)<0 \quad \left(s_0(h) < s \leq \frac{1}{1+\sqrt{1-h}}\right) \tag{7.39}$$

by (7.33), (7.34) and (7.35). Thus we complete the proof. □

Lemma 7.6 *Let $h \in (h_0,\ 1)$ be fixed. Then, (6.30) in s has the unique solution.*

Proof. Let us fixed $h \in (h_0,\ 1)$. We have

$$\frac{\partial^3 F}{\partial s^3}(h,0)<0, \quad \frac{\partial^3 F}{\partial s^3}\left(h, \frac{1}{1+\sqrt{1-h}}\right)>0 \tag{7.40}$$

by (7.30) and (7.11) with $h_0 < h < 1$, respectively. Hence there exists the unique $\tilde{s_3}(h) \in (0, 1/(1+\sqrt{1-h}))$ such that

$$\frac{\partial^3 F}{\partial s^3}(h,s)<0 \quad (0 \leq s < \tilde{s_3}(h)), \tag{7.41}$$

$$\frac{\partial^3 F}{\partial s^3}(h,\tilde{s_3}(h))=0, \tag{7.42}$$

$$\frac{\partial^3 F}{\partial s^3}(h,s)>0 \quad \left(\tilde{s_3}(h) < s \leq \frac{1}{1+\sqrt{1-h}}\right) \tag{7.43}$$

by (7.25).

On the other hand, we get

$$\frac{\partial^2 F}{\partial s^2}(h,0) > 0, \quad \frac{\partial^2 F}{\partial s^2}\left(h, \frac{1}{1+\sqrt{1-h}}\right) < 0 \tag{7.44}$$

by (7.23) and (7.10), respectively. Thus there exists the unique $\tilde{s_2}(h) \in (0, 1/(1+\sqrt{1-h}))$ such that

$$\frac{\partial^2 F}{\partial s^2}(h,s) > 0 \quad (0 \leq s < \tilde{s_2}(h)), \tag{7.45}$$

$$\frac{\partial^2 F}{\partial s^2}(h,\tilde{s_2}(h)) = 0, \tag{7.46}$$

$$\frac{\partial^2 F}{\partial s^2}(h,s) < 0 \quad \left(\tilde{s_2}(h) < s \leq \frac{1}{1+\sqrt{1-h}}\right) \tag{7.47}$$

by (7.41), (7.42) and (7.43).

Moreover, we obtain

$$\frac{\partial F}{\partial s}(h,0) < 0, \quad \frac{\partial F}{\partial s}\left(h, \frac{1}{1+\sqrt{1-h}}\right) > 0 \tag{7.48}$$

by (7.22) and (7.9), respectively. Therefore there exists the unique $\tilde{s_1}(h) \in (0, 1/(1+\sqrt{1-h}))$ such that

$$\frac{\partial F}{\partial s}(h,s) < 0 \quad (0 \leq s < \tilde{s_1}(h)), \tag{7.49}$$

$$\frac{\partial F}{\partial s}(h,\tilde{s_1}(h)) = 0, \tag{7.50}$$

$$\frac{\partial F}{\partial s}(h,s) < 0 \quad \left(\tilde{s_1}(h) < s \leq \frac{1}{1+\sqrt{1-h}}\right) \tag{7.51}$$

by (7.45), (7.46) and (7.47).

Now, we have

$$F(h,0) > 0, \quad F\left(h, \frac{1}{1+\sqrt{1-h}}\right) < 0 \tag{7.52}$$

due to (6.27) and (6.28). Consequently, there exists the unique $\tilde{s_0}(h)$ such that

$$F(h,s) > 0 \quad (0 \leq s < \tilde{s_0}), \tag{7.53}$$

$$F(h,\tilde{s_0}) = 0, \tag{7.54}$$

$$F(h,s) < 0 \quad \left(\tilde{s_0} < s \leq \frac{1}{1+\sqrt{1-h}}\right) \tag{7.55}$$

by (7.49), (7.50) and (7.51). Thus we complete the proof. □

Proof of Proposition 6.6. We obtain conclusions by Lemmas 7.5 and 7.6.□

7.2 Lemmas for proof of Lemma 7.2

We prepare several lemmas to prove Lemma 7.2.

Lemma 7.7 *Let $g_2(H,u)$ be defined by (7.16). Then*

$$g_2\left(H,\ \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) > 0 \quad \left(0 < H \leq \frac{31}{100}\right). \tag{7.56}$$

Proof. We note that the graphs of $g_2(h,u)$ is like Figure 7.1.

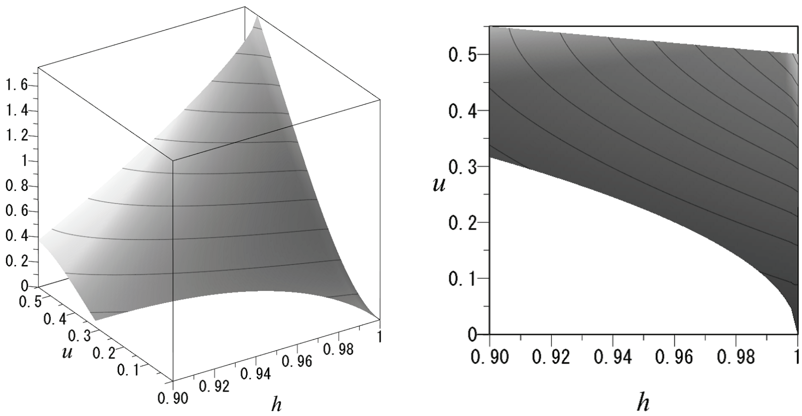

Figure 7.1: Graphs of $g_2(h,u)$ for $9039/10000 < h < 1$, $\sqrt{1-h} < u < 1-h/2$.

We may show that

$$g_2(H,u) > 0 \quad \left(0 < H \leq \frac{31}{100},\quad H < u < \frac{1+H^2}{2}\right) \tag{7.57}$$

by Lemma 2.1.

We have

$$\begin{aligned}
g_2(H,H) &= 2H^2(6H^4 - 13H^3 - 10H^2 - 13H + 6)(1-H)^2 > 0,\\
g_{2,u}(H,u) &= -6\left(H^2+1\right)\left(5H^2+3H+5\right)u^2\\
&\quad + \left(24H^6 - 8H^5 + 74H^4 + 44H^3 + 74H^2 - 8H + 24\right)u\\
&\quad - 2H^2\left(18H^4 + H^3 + 8H^2 + H + 18\right),\\
g_{2,u}(H,H) &= 2H\left(12H^6 - 37H^5 + 27H^4 - 16H^3 + 27H^2 - 37H + 12\right) > 0,\\
g_{2,u}&\left(H, \frac{1+H^2}{2}\right)\\
&= \frac{1}{2}\left(9H^6 - 35H^5 + 27H^4 - 14H^3 + 27H^2 - 35H + 9\right)(H+1)^2 > 0
\end{aligned}$$

for $0 < H \leq 31/100$ by virtue of Sturm's theorem. Hence we obtain (7.57). Thus we complete the proof. □

Lemma 7.8 *It holds that*

$$\frac{d}{dH}\left(\frac{K(\sqrt{1-H^2})^3}{(1-H^2)^2}\cdot g_3\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right)\right) < 0 \quad (0 < H < 1). \quad (7.58)$$

Proof. We have

$$\begin{aligned}&\frac{d}{dh}\left(\frac{K(\sqrt{1-H^2})^3}{(1-H^2)^2}\cdot g_3\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right)\right)\\ &= -\frac{K(\sqrt{1-H^2})^3}{H(1-H)^3(1+H)^3}\cdot f\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right), \end{aligned}\quad (7.59)$$

where

$$\begin{aligned} f(H,u) &:= c_3(H)\cdot u^3 + c_2(H)\cdot u^2 + c_1(H)\cdot u \\ &\quad + H^4\left(21\,H^4 - 6\,H^3 - 133\,H^2 - 24\,H - 30\right), \quad (7.60)\\ c_1(H) &:= -3\,H^2\left(18\,H^6 - 8\,H^5 - 63\,H^4 + 24\,H^3 - 45\,H^2 - 14\,H - 20\right) \quad (7.61)\\ c_2(H) &:= 3\,H\left(8\,H^7 - 8\,H^6 - 7\,H^5 + 34\,H^4 - 73\,H^3 - 44\,H^2 - 54\,H + 4\right), \quad (7.62)\\ c_3(H) &:= \left(-40\,H^6 - 48\,H^5 + 87\,H^4 + 120\,H^3 + 135\,H^2 + 6\,H + 8\right). \quad (7.63)\end{aligned}$$

We may show that

$$f(H,u) > 0 \quad (0 < H < 1, \quad H < u < 1) \quad (7.64)$$

by Lemma 2.1.

We obtain

$$f(H,H) = 10H^3(12H^5 - 35H^4 - 94H^3 + 33H^2 + 8H + 40)(1-H)^2 > 0,$$

and $c_3(H) > 0$ for $H \in (0,\ 1)$ by Sturm's theorem. Moreover, we get

$$f_u(H,u) = 3c_3(H)\cdot u^2 + 2c_2(H)\cdot u + c_1(H) > 0 \quad (0 < H < 1),$$

since

$$\begin{aligned}&c_2(H)^2 - 3c_3(H)\cdot c_1(H)\\ &= -9\,H^2\,(H+1)^3\left(-64\,H^{11} + 320\,H^{10} - 1008\,H^8 - 103\,H^7\right.\\ &\left.-183\,H^6 + 3211\,H^5 - 1837\,H^4 + 1004\,H^3 - 548\,H^2 + 232\,H + 144\right) < 0,\end{aligned}$$

for $0 < H < 1$ by virtue of Sturm's theorem.

Therefore, we get (7.64). Thus we complete the proof. □

Lemma 7.9 *There exists the unique* $h_0 \in [1-(31/100)^2, 1-(1/10)^2]$ *such that*

$$g_3\left(\sqrt{1-h}, \frac{E(\sqrt{h})}{K(\sqrt{h})}\right) < 0 \quad (0 < h < h_0), \tag{7.65}$$

$$g_3\left(\sqrt{1-h}, \frac{E(\sqrt{h})}{K(\sqrt{h})}\right) = 0, \tag{7.66}$$

$$g_3\left(\sqrt{1-h}, \frac{E(\sqrt{h})}{K(\sqrt{h})}\right) > 0 \quad (h_0 < h < 1). \tag{7.67}$$

Proof. We have

$$g_3\left(\frac{1}{10},\ E\left(\sqrt{1-\left(\frac{1}{10}\right)^2}\right) \Big/ K\left(\sqrt{1-\left(\frac{1}{10}\right)^2}\right)\right) > 0,$$

since it holds that

$$\frac{1}{10} < E\left(\sqrt{1-\left(\frac{1}{10}\right)^2}\right) \Big/ K\left(\sqrt{1-\left(\frac{1}{10}\right)^2}\right) < \frac{121}{400} = \frac{1}{4}\left(1+\frac{1}{10}\right)^2$$

by Lemma 2.1, and

$$g_3\left(\frac{1}{10}, u\right)$$
$$= -\frac{45349u^3}{2500} + \frac{1855347u^2}{250000} - \frac{194049u}{500000} + \frac{1777}{1000000} > 0 \quad \left(\frac{1}{10} < u < \frac{121}{400}\right)$$

by Sturm's theorem.

We have

$$g_3\left(\frac{31}{100},\ E\left(\sqrt{1-\left(\frac{31}{100}\right)^2}\right) \Big/ K\left(\sqrt{1-\left(\frac{31}{100}\right)^2}\right)\right) < 0,$$

since it holds that

$$\frac{31}{100} < E\left(\sqrt{1-\left(\frac{31}{100}\right)^2}\right) \Big/ K\left(\sqrt{1-\left(\frac{31}{100}\right)^2}\right) < 1$$

by Lemma 2.1, and

$$g_3\left(\frac{31}{100}, u\right)$$
$$= -\frac{316740017\,u^3}{12500000} + \frac{1501548449781\,u^2}{125000000000}$$
$$- \frac{2033076415239\,u}{500000000000} + \frac{189263623177}{1000000000000} < 0 \quad \left(\frac{31}{100} < u < 1\right)$$

by Sturm's theorem.

Thus, there exists the unique $H_0 \in (1/10, 31/100)$ such that

$$g_3\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) > 0 \quad (0 < h < H_0),$$
$$g_3\left(H_0, \frac{E(\sqrt{1-H_0^2})}{K(\sqrt{1-H_0^2})}\right) = 0,$$
$$g_3\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) < 0 \quad (H_0 < h < 1).$$

Consequently we complete the proof by putting $h := 1 - H^2$ and $h_0 := 1 - H_0^2$.□

Lemma 7.10 *Let $g_4(H, u)$ be defined by (7.18). Then*

$$g_4\left(H, \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) > 0 \quad \left(\frac{1}{10} \leq H < 1\right). \tag{7.68}$$

Proof. We note that the graphs of $g_4(h, u)$ is like Figure 7.2.

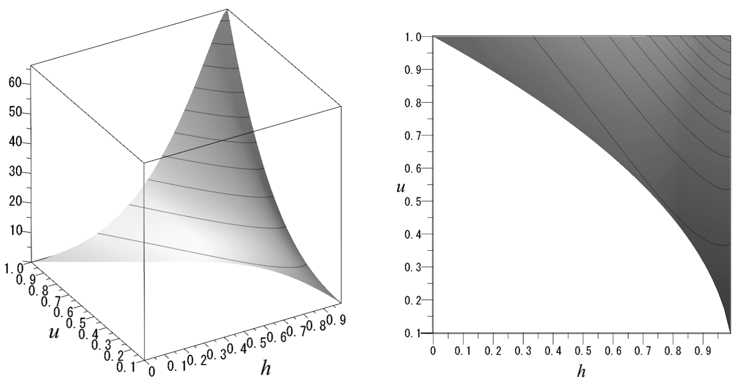

Figure 7.2: Graphs of $g_4(h, u)$ for $0 < h < 99/100$, $\sqrt{1-h} < u < 1 - h/2$.

We may show that

$$g_4(H, u) > 0 \quad \left(\frac{1}{10} \leq H < 1, \quad H < u < \frac{1+H^2}{2}\right) \tag{7.69}$$

by Lemma 2.1.

We have

$$
\begin{aligned}
g_4(H,H) &= 2H^2(-4H^4+101H^3+140H^2+101H-4)(1-H)^4 > 0,\\
g_{4,u}(H,H) &= 2H(-8H^6+205H^5-27H^4-60H^3-27H^2+205H-8)\\
&\quad \cdot (1-H)^2 > 0,\\
g_{4,uu}(H,u) &= 2\left(216\,H^6-48\,H^5-312\,H^3-48\,H+216\right)u\\
&\quad -16\,H^8+128\,H^7-658\,H^6+296\,H^5+516\,H^4\\
&\quad +296\,H^3-658\,H^2+128\,H-16,\\
g_{4,uu}(H,H) &= -16H^8+560H^7-754H^6+296H^5-108H^4\\
&\quad +296H^3-754H^2+560H-16>0,\\
g_{4,uu}&\left(H,\frac{1+H^2}{2}\right)\\
&= 2(100H^6-160H^5-H^4+130H^3-H^2-160H+100)(H+1)^2>0
\end{aligned}
$$

for $1/10 \le H < 1$ by virtue of Sturm's theorem. Hence we obtain (7.69). Thus we complete the proof. □

Lemma 7.11 *Let $g_5(H,u)$ be defined by (7.19). Then*

$$
g_5\left(H,\ \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) > 0 \quad (0<H<1)\,. \tag{7.70}
$$

Proof. We note that the graphs of $g_5(h,u)$ is like Figure 7.3.

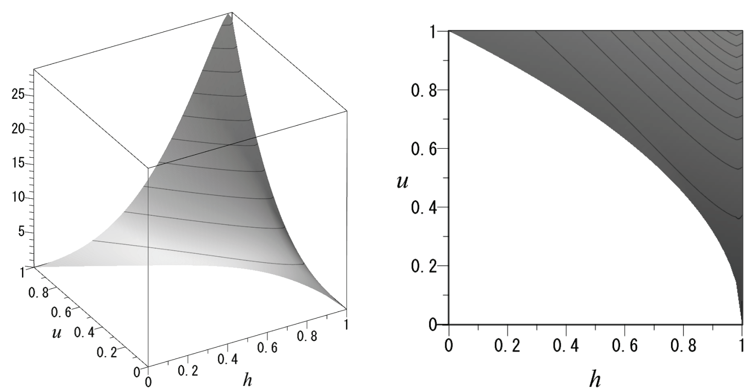

Figure 7.3: Graphs of $g_5(h,u)$.

We may show that

$$
g_5(H,u) > 0 \quad (0<H<1, \quad H<u<1) \tag{7.71}
$$

by Lemma 2.1.

We have

$$
\begin{aligned}
g_5(H,H) &= 2H^3(27H^2+40H+27)(1-H)^4 > 0,\\
g_{5,u}(H,H) &= 2H^2(59H^4+19H^3+4H^2+19H+59)(1-H)^2 > 0,\\
g_{5,uu}(H,u) &= 2\left(84\,H^6+36\,H^5-36\,H^4-144\,H^3-36\,H^2+36\,H+84\right)u\\
&\quad +2\,H\left(8\,H^6-95\,H^5+36\,H^4+110\,H^3+36\,H^2-95\,H+8\right),\\
g_{5,uu}(H,H) &= 2H(92H^6-59H^5-34H^3-59H+92) > 0,\\
g_{5_u u}(H,1) &= 2(H+1)(8H^6-19H^5+91H^4-17H^3-91H^2-40H+84) > 0
\end{aligned}
$$

for $0 < H < 1$ by virtue of Sturm's theorem. Hence we obtain (7.71). Thus we complete the proof. □

Lemma 7.12 *Let $g_6(H,u)$ be defined by (7.20). Then,*

$$g_6\left(H,\ \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right) > 0 \quad (0 < H < 1)\,. \tag{7.72}$$

Proof. We note that the graphs of $g_6(h,u)$ is like Figure 7.4.

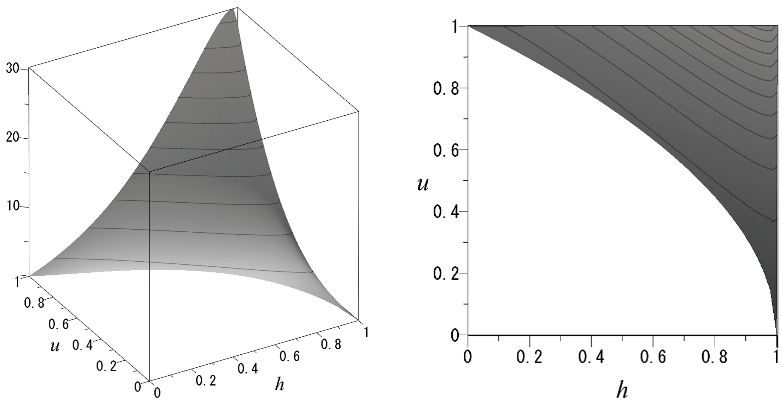

Figure 7.4: Graphs of $g_6(h,u)$.

We may show that

$$g_6(H,u) > 0 \quad (0 < H < 1,\quad H < u < 1) \tag{7.73}$$

by Lemma 2.1.

We have

$$
\begin{aligned}
g_6(1-H^2,u) &= 2H^3(17H^4-H^3-8H^2-H+17)(1-H)^2>0,\\
g_{6,u}(H,H) &= 2H^2(45H^6-38H^5-2H^3-38H+45)>0,\\
g_{6,uu}(H,u) &= 2\left(84\,H^6+84\,H^5-36\,H^4-156\,H^3-36\,H^2+84\,H+84\right)u\\
&\quad -2\,H^2\left(65\,H^4-17\,H^3-80\,H^2-17\,H+65\right),\\
g_{6,uu}(H,H) &= 2H(84H^6+19H^5-19H^4-76H^3-19H^2+19H+84)>0,\\
g_{6,uu}(H,1) &= 2(H+1)(19H^5+82H^4-38H^3-101H^2+84)>0
\end{aligned}
$$

for $0<H<1$ by virtue of Sturm's theorem. Hence we obtain (7.73). Thus we complete the proof. □

Lemma 7.13 *Let $g_7(H,u)$ be defined by (7.21). Then*

$$
g_7\left(H,\ \frac{E(\sqrt{1-H^2})}{K(\sqrt{1-H^2})}\right)>0 \quad (0<H<1)\,. \tag{7.74}
$$

Proof. We note that the graphs of $g_7(h,u)$ is like Figure 7.5.

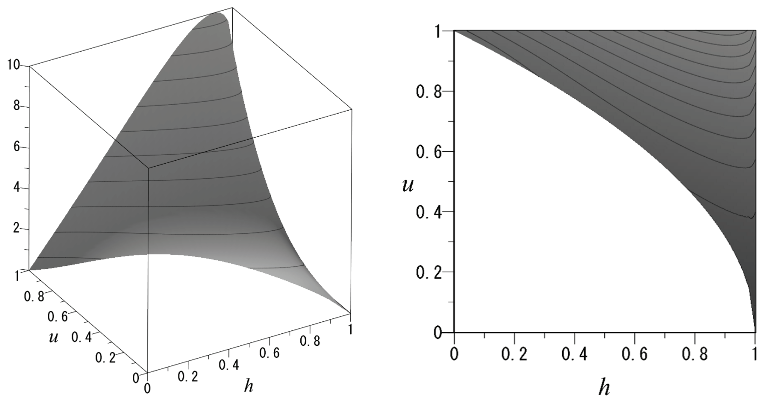

Figure 7.5: Graphs of $g_7(h,u)$.

We may show that

$$
g_7(H,u)>0 \quad (0<H<1,\quad H<u<1) \tag{7.75}
$$

by Lemma 2.1.

We have

$$
\begin{aligned}
g_7(H,u) &= 2H^3(4H^4+5H^3+6H^2+5H+4)(1-H)^2 > 0,\\
g_{7,u}(H,H) &= 6H^2(4H^6+H^5-H^4-4H^3-H^2+H+4) > 0,\\
g_{7,uu}(H,u) &= 24\left(2H^2+3H+2\right)\left(H^4-H^2+1\right)u\\
&\quad -6H^2\left(4H^4+H^3-2H^2+H+4\right),\\
g_{7,uu}(H,H) &= 6H(8H^6+8H^5-H^4-10H^3-H^2+8H+8) > 0,\\
g_{7,uu}(H,1) &= 6(H+1)(4H^5+7H^4-5H^3-8H^2+4H+8) > 0
\end{aligned}
$$

for $0 < H < 1$ by virtue of Sturm's theorem. Hence we obtain (7.75). Thus we complete the proof. □

8 Proof of Theorem 3.8

In this section, we show the existence of secondary bifurcation point, and give a proof of Theorem 3.8.

We note that $\boldsymbol{m}(1,\varepsilon^2) = 2$ $(0 < \varepsilon^2 < 1/\pi^2)$. We see that

$$\left.\frac{\partial \boldsymbol{m}(\tilde{V},\varepsilon^2)}{\partial \tilde{V}}\right|_{\tilde{V}=1} = 0 \quad \left(0 < \varepsilon^2 < \frac{1}{\pi^2}\right) \tag{8.1}$$

is equivalent to

$$\left.\frac{\partial \boldsymbol{m}(\tilde{V}(h,s),\varepsilon^2(h,s))}{\partial h}\right|_{s=\frac{1}{1+\sqrt{1-h}}} = 0 \quad (0 < h < 1)\,, \tag{8.2}$$

which implies

$$\frac{1}{2(1+\sqrt{1-h})(2-h)^{3/2}}\left(\frac{2-h}{1-h}\cdot\frac{E(\sqrt{h})}{K(\sqrt{h})} - 8\right) = 0 \quad (0 < h < 1)\,. \tag{8.3}$$

It holds that (8.3) has the unique solution $h = h_*$ with $h_* = 0.952\cdots$, and

$$\boldsymbol{m}(\tilde{V}(h_*,s_*),\varepsilon^2(h_*,s_*)) = 2, \tag{8.4}$$

$$\boldsymbol{m}_h(\tilde{V}(h_*,s_*),\varepsilon^2(h_*,s_*)) = 0, \tag{8.5}$$

$$\boldsymbol{m}_s(\tilde{V}(h_*,s_*),\varepsilon^2(h_*,s_*)) = 0. \tag{8.6}$$

Here, $s_* = 1/(1+\sqrt{1-h_*}) = 0.821\cdots$ and $\varepsilon_*^2 = \varepsilon^2(h_*,s_*) = (0.2353\cdots)^2 = 0.055\cdots$. By simple computation, we get

$$\boldsymbol{m}_{hh}(\tilde{V}(h_*,s_*),\varepsilon^2(h_*,s_*)) = \text{Pos.Const.}\cdot\left(5h_*^2 - 12h_* - 12\right) > 0, \tag{8.7}$$

$$\boldsymbol{m}_{ss}(\tilde{V}(h_*,s_*),\varepsilon^2(h_*,s_*)) = \text{Const.}\cdot\left(\frac{2-h_*}{1-h_*}\cdot\frac{E(\sqrt{h_*})}{K(\sqrt{h_*})} - 8\right) = 0, \tag{8.8}$$

$$\boldsymbol{m}_{sss}(\tilde{V}(h_*,s_*),\varepsilon^2(h_*,s_*)) > 0. \tag{8.9}$$

Hence we see from Taylor's expansion at $h = h_*$ that

$$\boldsymbol{m}(\tilde{V}(h,s_*),\varepsilon^2(h,s_*)) > 2 \tag{8.10}$$

for $h > h_*$ sufficiently close to h_*. On the other hands, we obtain from Taylor's expansion at $s = s_*$ that

$$\boldsymbol{m}(\tilde{V}(h_*,s),\varepsilon^2(h_*,s)) < 2 \tag{8.11}$$

for $s < s_*$ sufficiently close to s_*.

Therefore, we see that there exist ℓ_1 and ℓ_2 with $0 < \ell_1 < \ell_2$ such that

$$\boldsymbol{m}(\tilde{V}, \ell_1(\tilde{V}-1)+\varepsilon_*^2) < 2 \quad (\tilde{V} < 1 \text{ sufficiently close to } 1) \tag{8.12}$$

and

$$\boldsymbol{m}(\tilde{V}, \ell_2(\tilde{V}-1)+\varepsilon_*^2) > 2 \quad (\tilde{V} < 1 \text{ sufficiently close to } 1). \tag{8.13}$$

By Theorem 3.4 (ii), we obtain

$$\varepsilon^2(\tilde{V}) \to \varepsilon_*^2 \quad \text{as} \quad \tilde{V} \uparrow 1. \tag{8.14}$$

Therefore, it following from (1.12), that

$$\varepsilon^2(\tilde{V}) \to \varepsilon_*^2 \quad \text{as} \quad \tilde{V} \downarrow 1. \tag{8.15}$$

We explain the meaning of the above argument intuitively. Figure 8.1 shows graphs of $\boldsymbol{m}(\tilde{V}, \varepsilon^2)$ for each fixed ε^2.

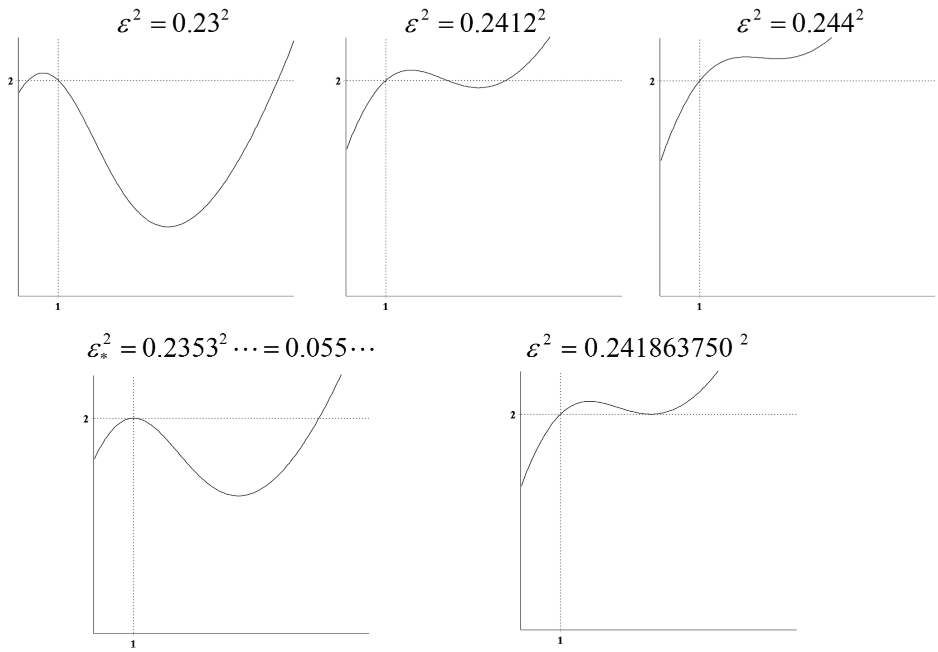

Figure 8.1: $\boldsymbol{m}(\tilde{V}, \varepsilon^2)$ for each ε^2

Figure 8.2 and 8.3 show curves in the parameter space, $h-s-m$ space, and corresponding curves in $\tilde{V}-\varepsilon^2-m$ space, respectively.

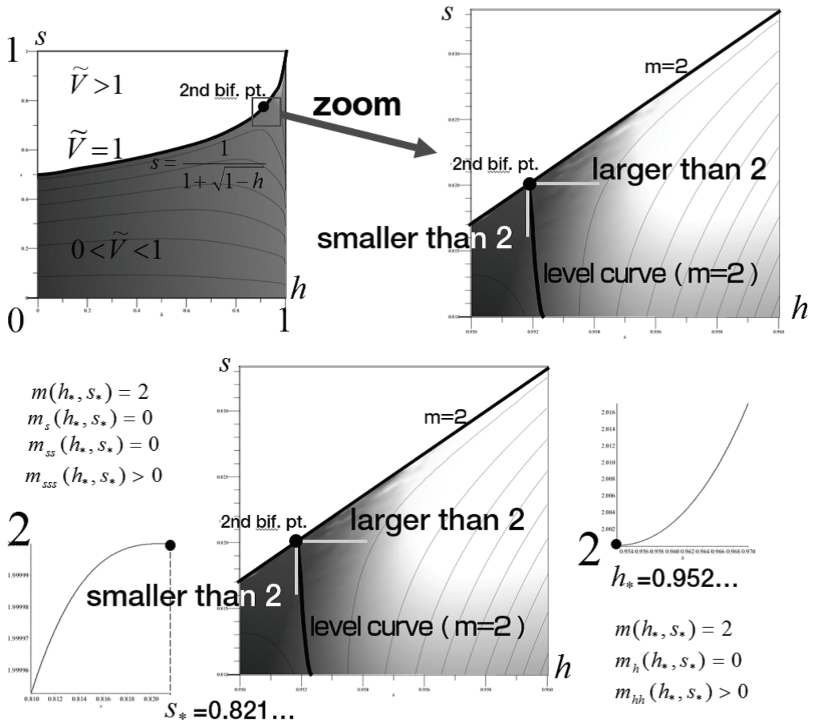

Figure 8.2: Secondary bifurcation point and curves in $h-s-m$ space

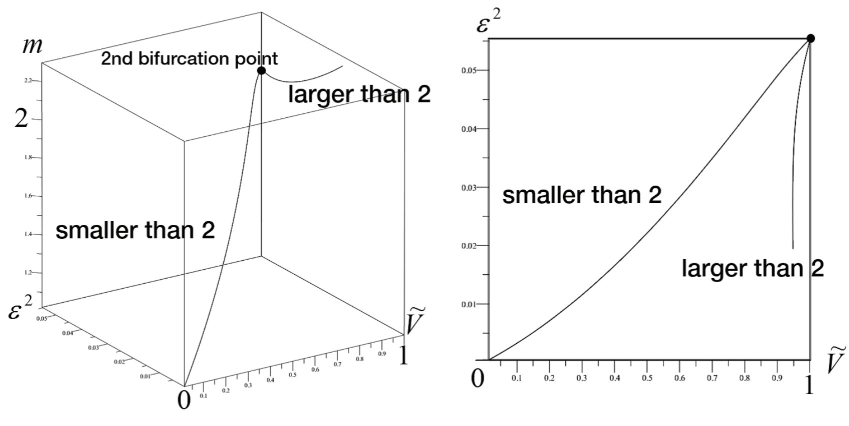

Figure 8.3: Secondary bifurcation point and curves in $\tilde{V}-\varepsilon^2-m$ space

9 Numerical results about the stability

Let us explain the stability of solutions of (SLP) observed by numerical computations. Let us see Figure 9.1. Stationary solutions corresponding to the points on the thick lines are locally stable and those on the broken lines are unstable.

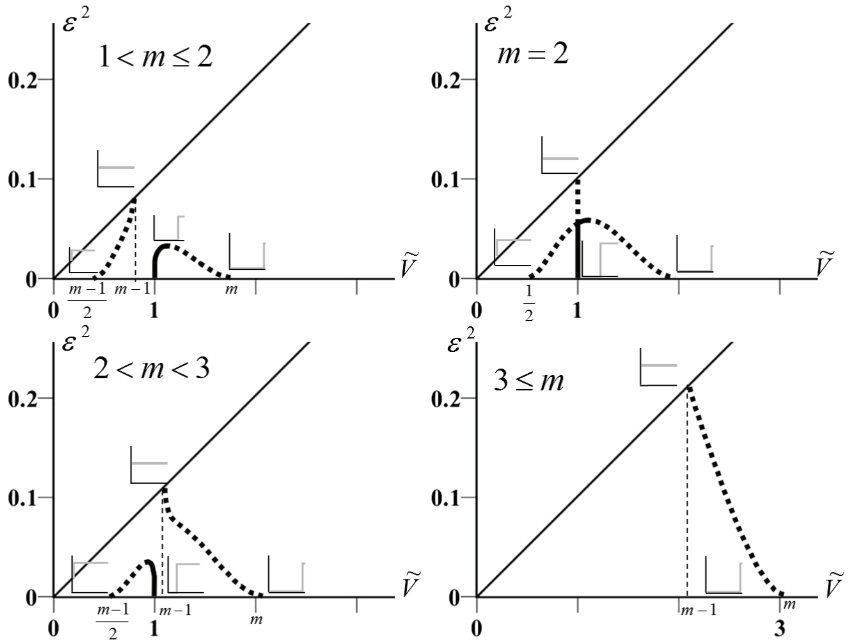

Figure 9.1: Stability of solutions of (SLP)

We show several figures to see the behavior of solutions of (TLP) in Figures 9.2-9.33. Left column are profiles of $W(x,t)$, and right column are profiles of $\tilde{V}(t)$.

Figures 9.2-9.9 show the case $m = 1.5$ and $\varepsilon = 0.0576$. Initial values are taken as $\tilde{V}(0) = 1.325581$, $W(x,0) = W(x; \tilde{V}(0), \varepsilon^2)$, where $W(x; \tilde{V}(0), \varepsilon^2)$ is a numerical computed solution of (SLP) with $m = 1.5$. It seems that $\tilde{V}(t) \to 1.000112$ and $W(x,t) \to W(x; 1.000112, \varepsilon^2)$ as $t \to \infty$ numerically, where $W(x; 1.000112, \varepsilon^2)$ is a numerical computed another solution of (SLP) on the same bifurcation curve with $m = 1.5$.

Figures 9.10-9.17 show the case $m = 2$ and $\varepsilon = 0.032854$. Initial values are taken as $\tilde{V}(0) = 0.53846$, $W(x,0) = W(x; \tilde{V}(0), \varepsilon^2)$, where $W(x; \tilde{V}(0), \varepsilon^2)$ is a numerical computed solution of (SLP) with $m = 2$. It seems that $\tilde{V}(t) \to 1$ and $W(x,t) \to W(x; 1, \varepsilon^2)$ as $t \to \infty$ numerically, where $W(x; 1, \varepsilon^2)$ is a numerical computed symmetric solution of (SLP).

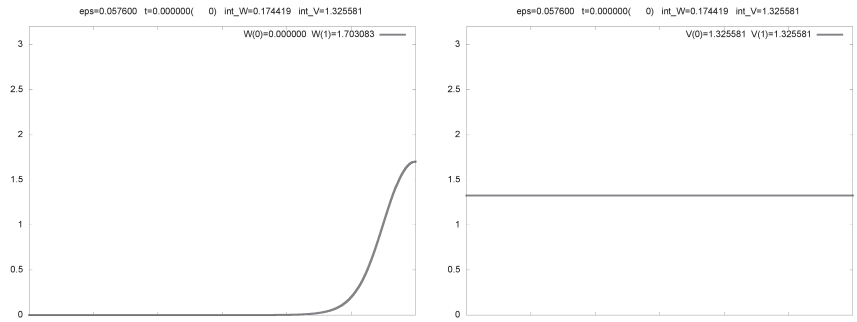

Figure 9.2: $m = 1.5$, $\tilde{V}(0) = 1.32558$, $\varepsilon = 0.0576$, $t = 0$

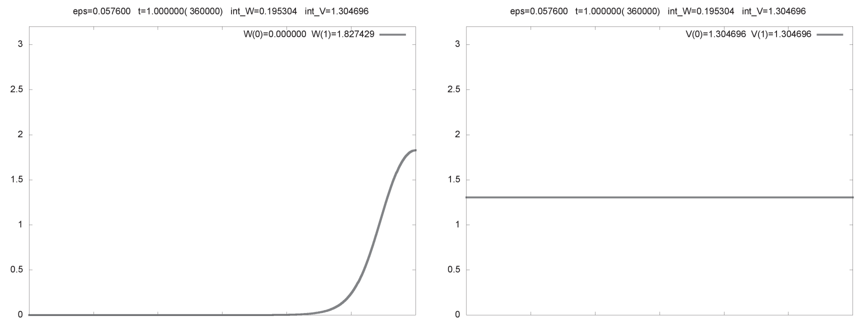

Figure 9.3: $m = 1.5$, $\tilde{V}(0) = 1.32558$, $\varepsilon = 0.0576$, $t = 1$

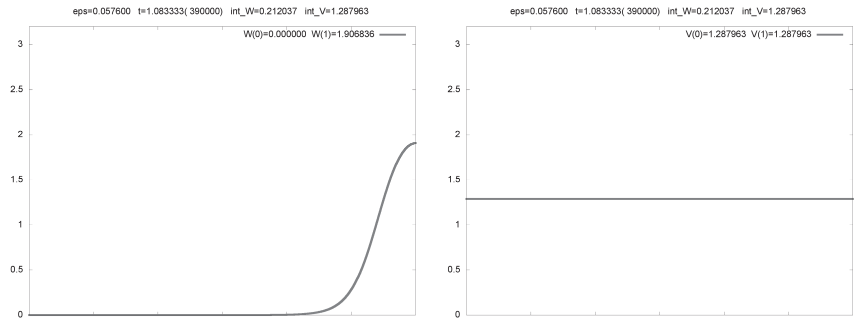

Figure 9.4: $m = 1.5$, $\tilde{V}(0) = 1.32558$, $\varepsilon = 0.0576$, $t = 1.083333$

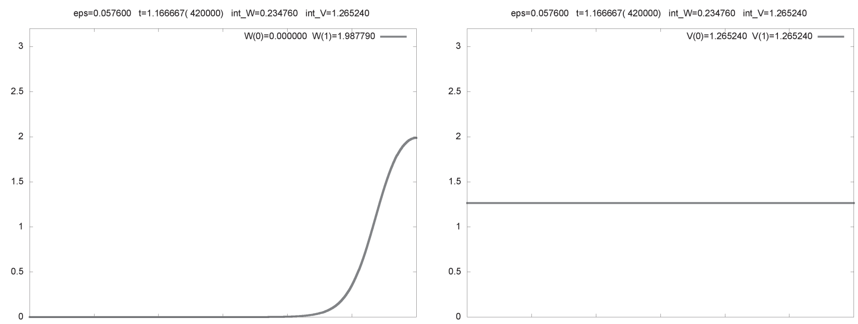

Figure 9.5: $m = 1.5$, $\tilde{V}(0) = 1.32558$, $\varepsilon = 0.0576$, $t = 1.166667$

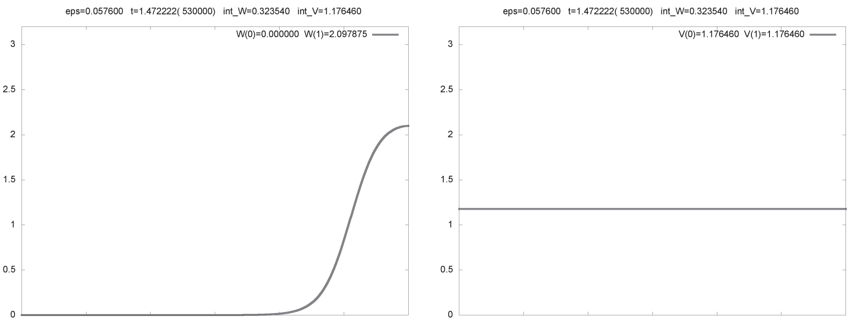

Figure 9.6: $m = 1.5$, $\tilde{V}(0) = 1.32558$, $\varepsilon = 0.0576$, $t = 1.472222$

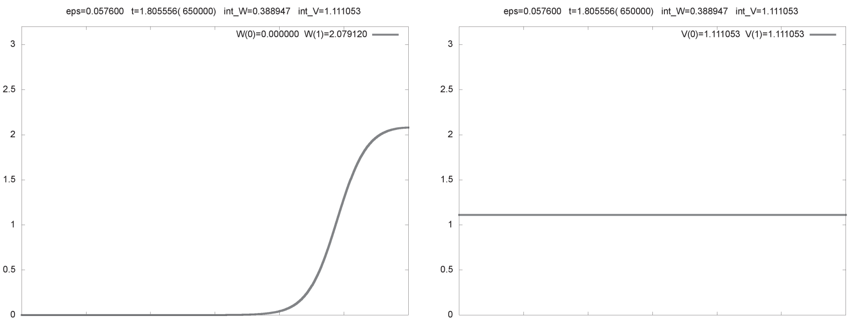

Figure 9.7: $m = 1.5$, $\tilde{V}(0) = 1.32558$, $\varepsilon = 0.0576$, $t = 1.805556$

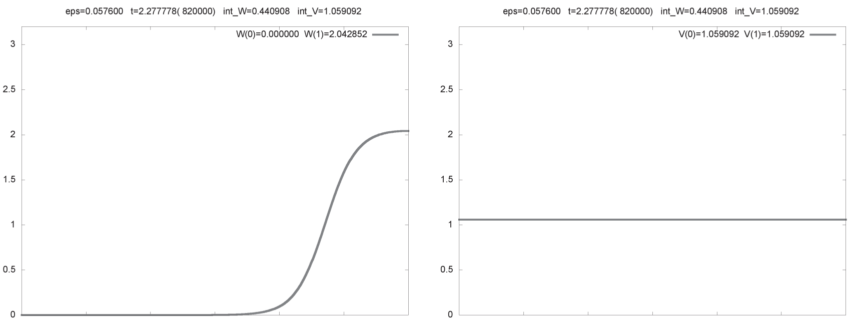

Figure 9.8: $m = 1.5$, $\tilde{V}(0) = 1.32558$, $\varepsilon = 0.0576$, $t = 2.277778$

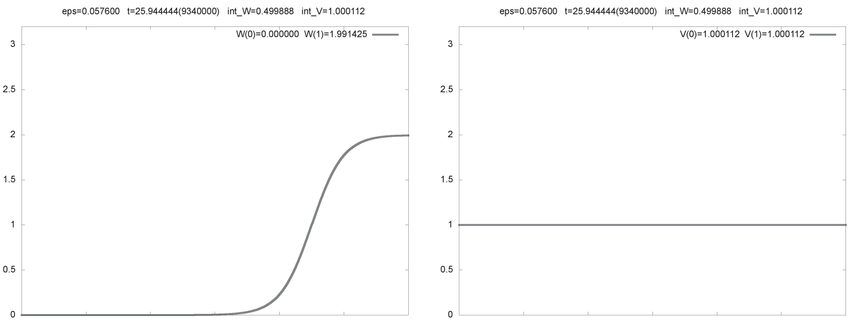

Figure 9.9: $m = 1.5$, $\tilde{V}(0) = 1.32558$, $\varepsilon = 0.0576$, $t = 25.944444$

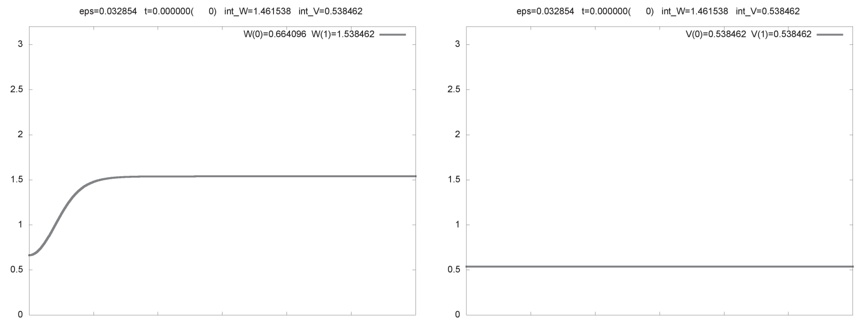

Figure 9.10: $m = 2$, $\tilde{V}(0) = 0.53846$, $\varepsilon = 0.032854$, $t = 0$

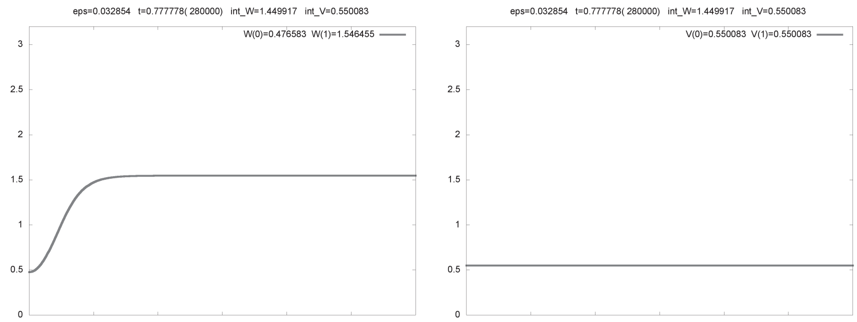

Figure 9.11: $m = 2$, $\tilde{V}(0) = 0.53846$, $\varepsilon = 0.032854$, $t = 0.777778$

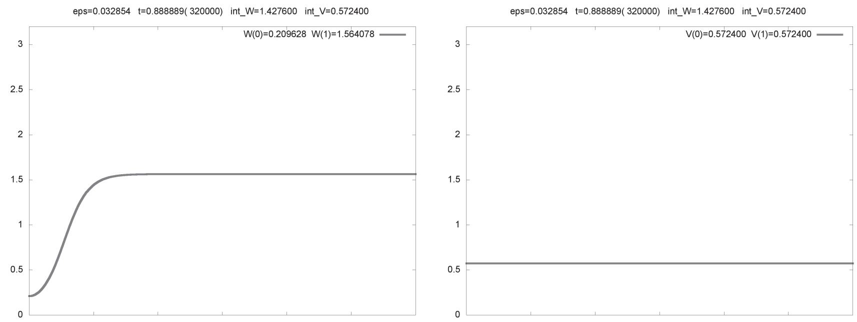

Figure 9.12: $m = 2$, $\tilde{V}(0) = 0.53846$, $\varepsilon = 0.032854$, $t = 0.888889$

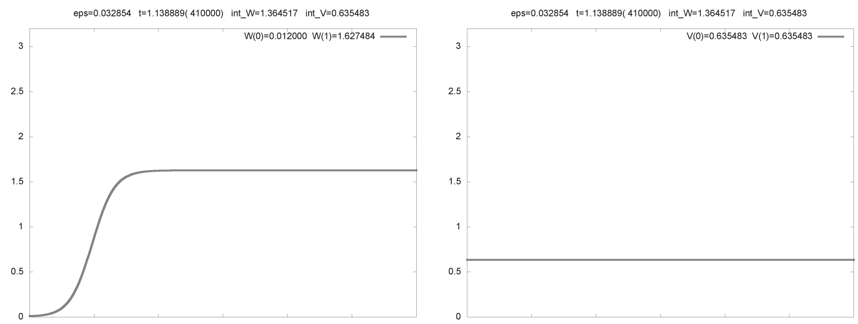

Figure 9.13: $m = 2$, $\tilde{V}(0) = 0.53846$, $\varepsilon = 0.032854$, $t = 1.138889$

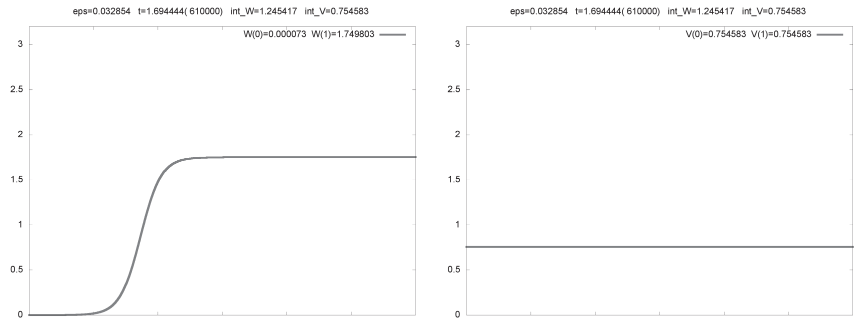

Figure 9.14: $m = 2$, $\tilde{V}(0) = 0.53846$, $\varepsilon = 0.032854$, $t = 1.694444$

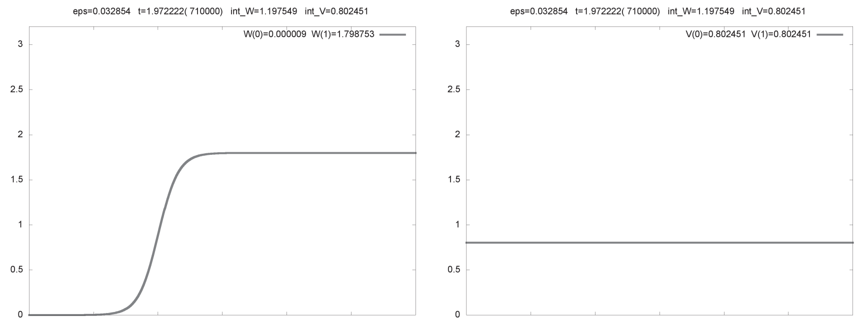

Figure 9.15: $m = 2$, $\tilde{V}(0) = 0.53846$, $\varepsilon = 0.032854$, $t = 1.972222$

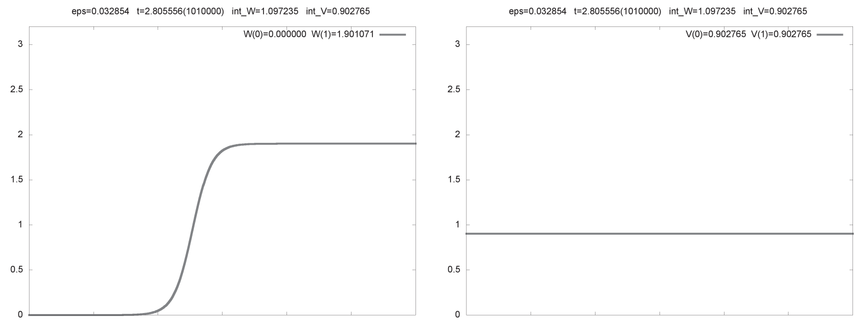

Figure 9.16: $m = 2$, $\tilde{V}(0) = 0.53846$, $\varepsilon = 0.032854$, $t = 2.805556$

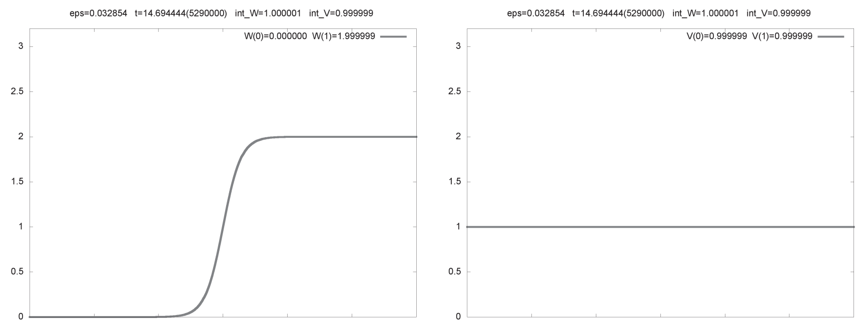

Figure 9.17: $m = 2$, $\tilde{V}(0) = 0.53846$, $\varepsilon = 0.032854$, $t = 14.694444$

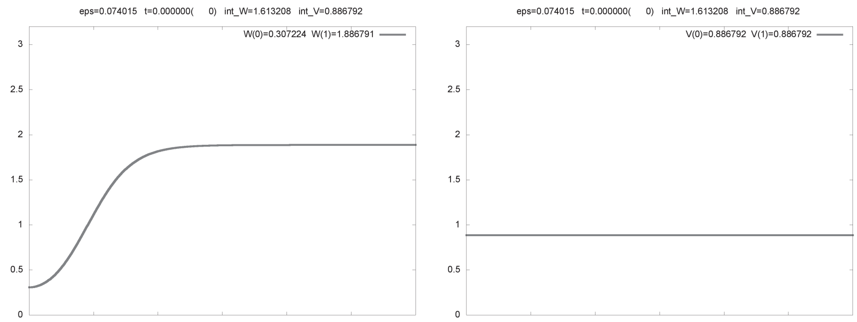

Figure 9.18: $m = 2.5$, $\tilde{V}(0) = 0.88679$, $\varepsilon = 0.074015$, $t = 0$

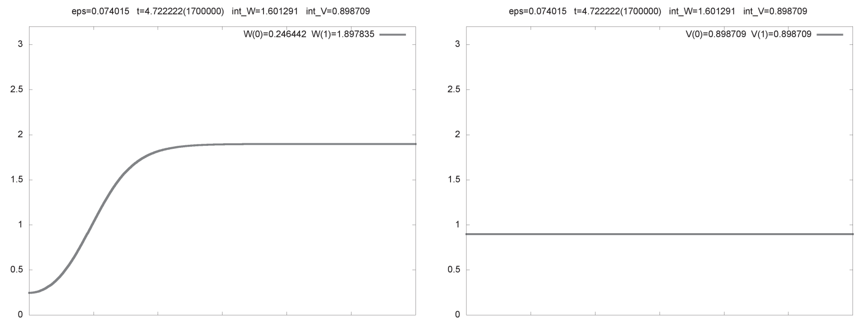

Figure 9.19: $m = 2.5$, $\tilde{V}(0) = 0.88679$, $\varepsilon = 0.074015$, $t = 4.722222$

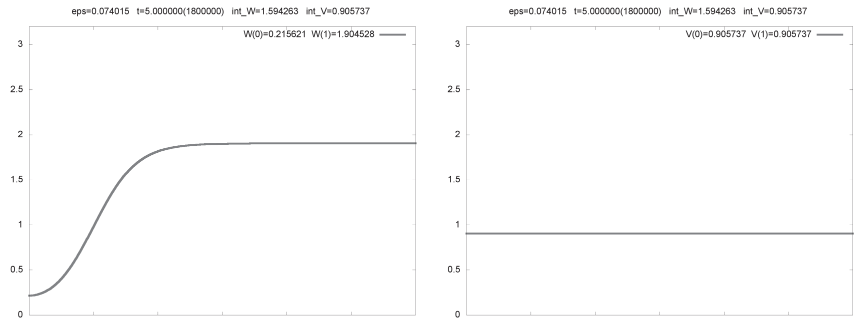

Figure 9.20: $m = 2.5$, $\tilde{V}(0) = 0.88679$, $\varepsilon = 0.074015$, $t = 5$

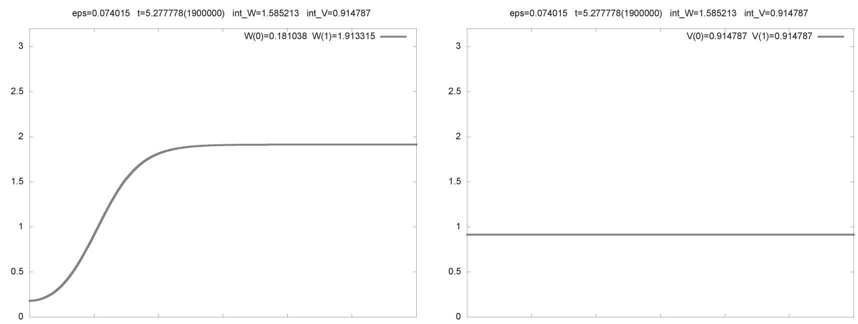

Figure 9.21: $m = 2.5$, $\tilde{V}(0) = 0.88679$, $\varepsilon = 0.074015$, $t = 5.277778$

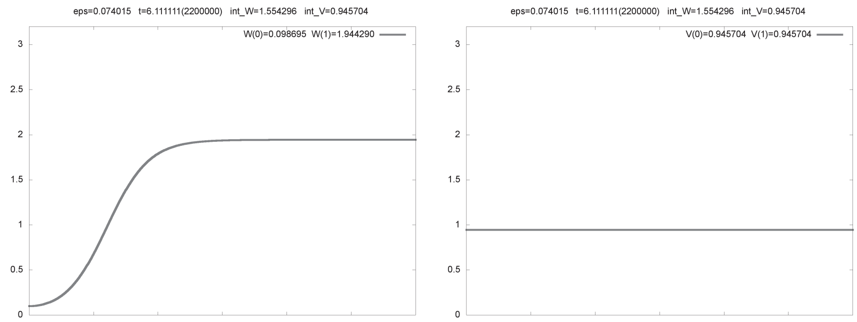

Figure 9.22: $m = 2.5$, $\tilde{V}(0) = 0.88679$, $\varepsilon = 0.074015$, $t = 6.111111$

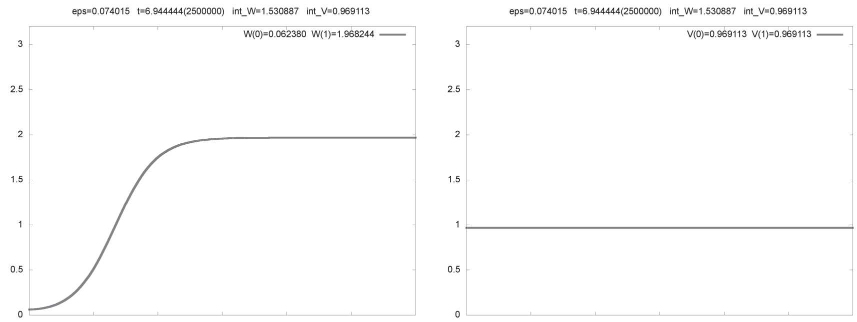

Figure 9.23: $m = 2.5$, $\tilde{V}(0) = 0.88679$, $\varepsilon = 0.074015$, $t = 6.944444$

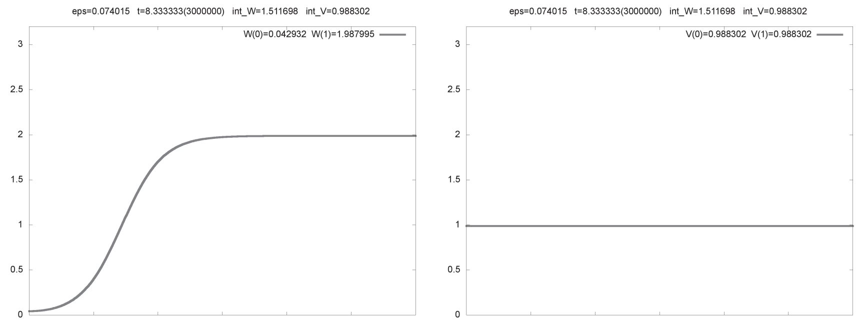

Figure 9.24: $m = 2.5$, $\tilde{V}(0) = 0.88679$, $\varepsilon = 0.074015$, $t = 8.333333$

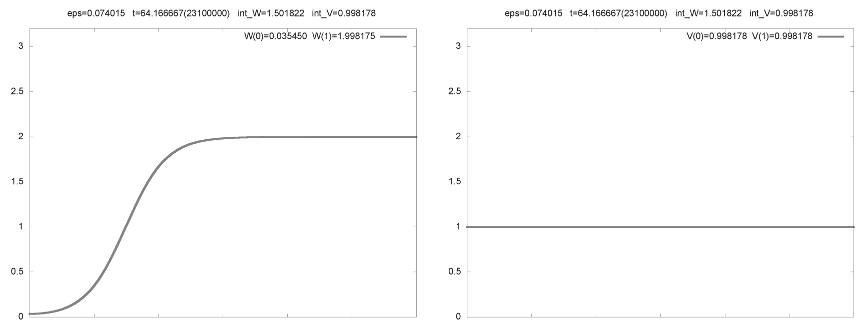

Figure 9.25: $m = 2.5$, $\tilde{V}(0) = 0.88679$, $\varepsilon = 0.074015$, $t = 64.166667$

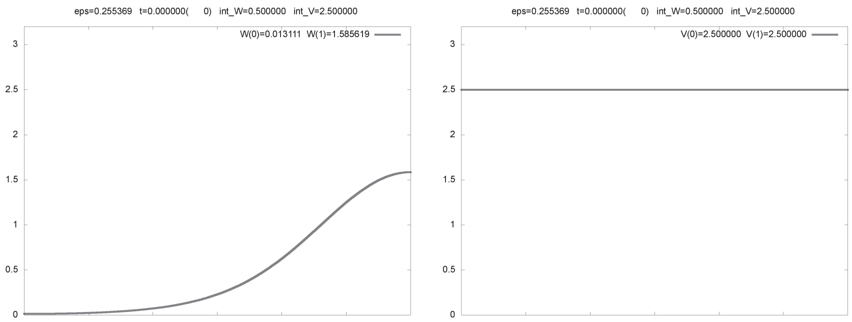

Figure 9.26: $m = 3$, $\tilde{V}(0) = 2.5$, $\varepsilon = 0.255369$, $t = 0$

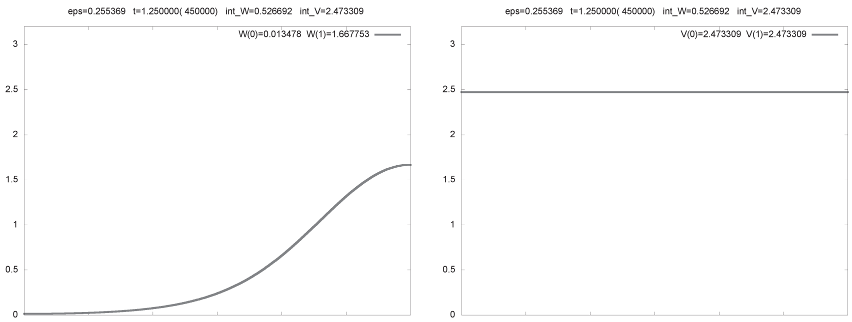

Figure 9.27: $m = 3$, $\tilde{V}(0) = 2.5$, $\varepsilon = 0.255369$, $t = 1.25$

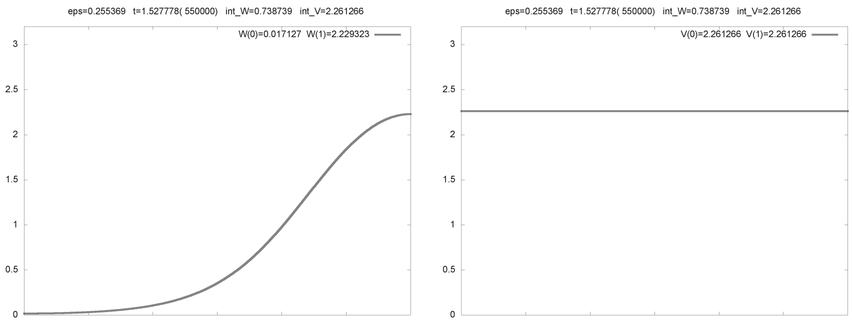

Figure 9.28: $m = 3$, $\tilde{V}(0) = 2.5$, $\varepsilon = 0.255369$, $t = 1.527778$

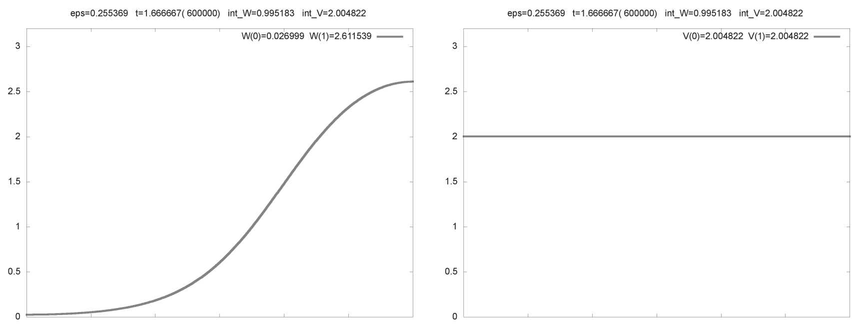

Figure 9.29: $m = 3$, $\tilde{V}(0) = 2.5$, $\varepsilon = 0.255369$, $t = 1.666667$

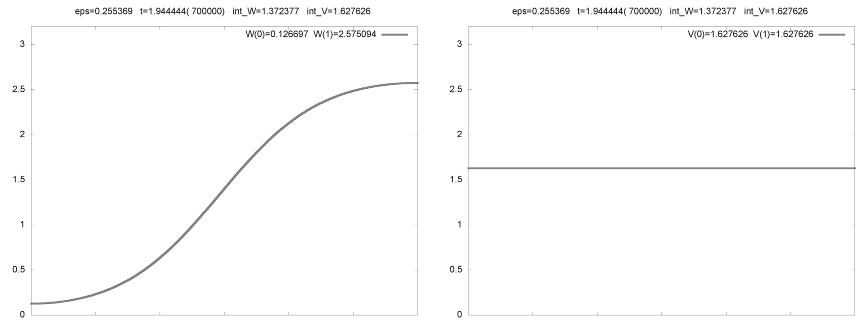

Figure 9.30: $m = 3$, $\tilde{V}(0) = 2.5$, $\varepsilon = 0.255369$, $t = 1.944444$

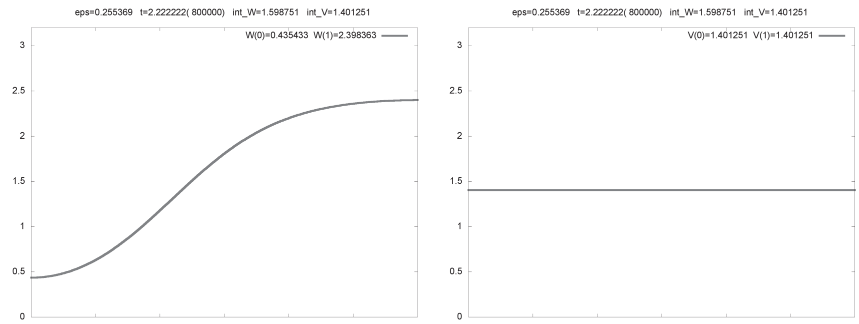

Figure 9.31: $m = 3$, $\tilde{V}(0) = 2.5$, $\varepsilon = 0.255369$, $t = 2.222222$

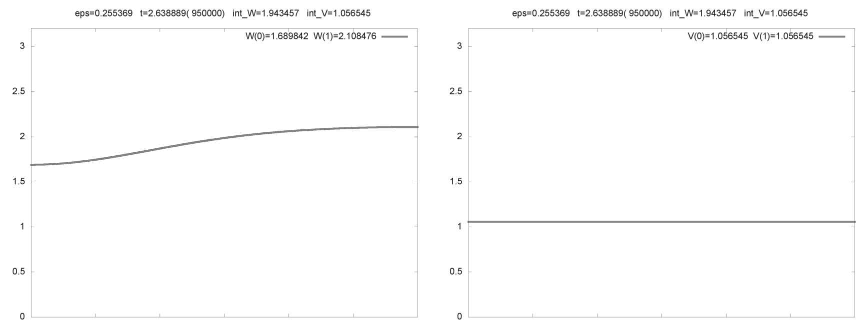

Figure 9.32: $m = 3$, $\tilde{V}(0) = 2.5$, $\varepsilon = 0.255369$, $t = 2.638889$

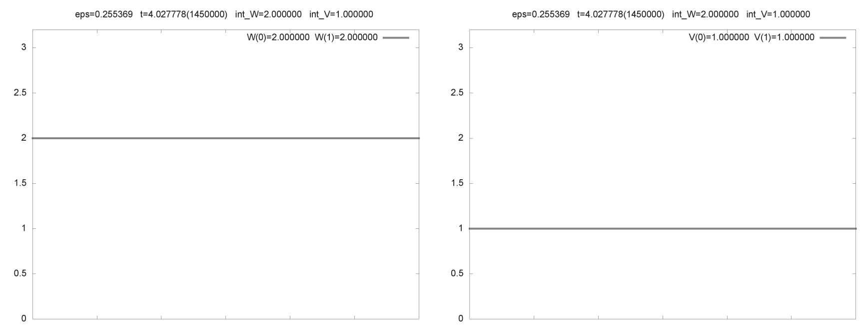

Figure 9.33: $m = 3$, $\tilde{V}(0) = 2.5$, $\varepsilon = 0.255369$, $t = 4.027778$

Figures 9.18-9.25 show the case m=2.5 and ε=0.074015. Initial values are taken as $\tilde{V}(0) = 0.886792$, $W(x, 0) = W(x; \tilde{V}(0), \varepsilon^2)$, where $W(x; \tilde{V}(0), \varepsilon^2)$ is a numerical computed solution of (SLP) with $m = 2.5$. It seems that $\tilde{V}(t) \to 0.998178$ and $W(x, t) \to W(x; 0.998178, \varepsilon^2)$ as $t \to \infty$ numerically, where $W(x; 0.998178, \varepsilon^2)$ is a numerical computed another solution of (SLP) on the same bifurcation curve with $m = 2.5$.

Figures 9.26-9.33 show the case $m = 3$ and $\varepsilon = 0.255369$. Initial values are taken as $\tilde{V}(0)$=2.5, $W(x, 0) = W(x; \tilde{V}(0), \varepsilon^2)$, where $W(x; \tilde{V}(0), \varepsilon^2)$ is a numerical computed solution of (SLP) with $m = 3$. It seems that $\tilde{V}(t) \to 1$ and $W(x, t) \to 2$ as $t \to \infty$ numerically.

10 Concluding remarks

In this paper we have clarified all global bifurcation curves for a cell polarization model. We have given answers of following problems:

- Existence and nonexistence of all global bifurcation curves for all given m?
- Direction and connection of bifurcation curves?
- Existence and uniqueness of the secondary bifurcation point?

Results are the first ones to clarify the existence and nonexistence of all global bifurcation curves including the unique existence of the secondary bifurcation point.

There are several things to be solved:

- Critical points of each global bifurcation curve.
- The stability.
- The global bifurcation curves of original cell polarization model for finite D.

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